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Weak Convex Domination in Graphs

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Abstract: *In a graph G = (V, E), a set D$\subset$ V is a weak convex set if $d_{<D>}(u,v) = d_G(u,v)$ for any two vertices u, v in D. A weak convex set is called as a weak convex dominating (WCD) set if each vertex of V-D is adjacent to at least one vertex in D. The weak convex domination number $\gamma_{wc}(G)$ is the smallest order of a weak convex dominating set of G and the codomination number of G, denoted by $\gamma_{wc}(\bar{G})$, is the weak convex domination number of its complement. In this paper, we found various bounds of these parameters and characterized the graphs, for which bounds are attained.*

Keywords: *domination number, distance, eccentricity, radius, diameter, self-centered, neighbourhood, weak convex set, weak convex dominating set.*

1. Introduction

Graphs discussed in this paper are undirected and simple. Unless otherwise stated the graphs which we consider are connected graphs only. For a graph G, let V(G) and E(G) denote its vertex and edge set respectively and p and q denote the cardinality of those sets respectively. The *degree* of a vertex *v* in a graph G is denoted by $\deg_G(v)$. The minimum and maximum degree in a graph is denoted by δ and Δ respectively. The length of any shortest path between any two vertices *u* and *v* of a connected graph G is called the *distance between u and v* and is denoted by $d_G(u, v)$. The distance between two vertices in different components of a disconnected graph is defined to be ∞. For a connected graph G, the *eccentricity* $e_G(v) = \max\{d_G(u, v): u \in V(G)\}$. If there is no confusion, we simply use the notion deg(*v*), d(*u*, *v*) and e(*v*) to denote degree, distance and eccentricity respectively for the concerned graph. The minimum and maximum eccentricities are the *radius* and *diameter* of G, denoted r(G) and diam(G) respectively. When these two are equal, the graph is called *self-centered* graph with radius r, equivalently is *r self-centered*. A vertex *u* is said to be an *eccentric vertex* of *v* in a graph G, if d(*u*, *v*) = e(*v*). In general, *u* is called an eccentric vertex, if it is an eccentric vertex of some vertex. For $v \in V(G)$, the *neighbourhood* $N_G(v)$ of *v* is the set of all vertices adjacent to *v* in G. The set

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$N_G[v] = N_G(v) \cup \{v\}$ is called the *closed neighbourhood* of v. A set S of edges in a graph is said to be *independent* if no two of the edges in S are adjacent. An edge e = (u, v) is a *dominating edge* in a graph G if every vertex of G is adjacent to at least one of u and v.

The concept of domination in graphs was introduced by Ore. A set $D \subseteq V(G)$ is called *dominating set* of G if every vertex in V(G)-D is adjacent to some vertex in D. D is said to be a *minimal* dominating set if D-$\{v\}$ is not a dominating set for any $v \in D$. The *domination number* γ(G) of G is the minimum cardinality of a dominating set. We call a set of vertices a γ-set if it is a dominating set with cardinality γ(G). Different types of dominating sets have been studied by imposing conditions on the dominating sets. A dominating set D is called *connected* (*independent*) dominating set if the induced subgraph <D> is connected (independent). D is called a *total dominating set* if every vertex in V(G) is adjacent to some vertex in D.

A cycle of D of a graph G is called a *dominating cycle* of G, if every vertex in V–D is adjacent to some vertex in D. A dominating set D of a graph G is called a *clique dominating set* of G if <D> is complete. A set D is called an *efficient dominating set* of G if every vertex in V – D is adjacent to exactly one vertex in D. A set $D \subseteq V$ is called a *global dominating set* if D is a dominating set in G and $\overline{G}$. A set D is called a *restrained dominating set* if every vertex in V(G)-D is adjacent to a vertex in D and another vertex in V(G)-D. By $\gamma_c, \gamma_i, \gamma_t, \gamma_o, \gamma_k, \gamma_e, \gamma_g$ and γ_r, we mean the minimum cardinality of a connected dominating set, independent dominating set, total dominating set, cycle dominating set, clique dominating set, efficient dominating set, global dominating set and restrained dominating set respectively.

In this paper, we introduce a new dominating set called weak convex dominating set of a graph through which we analyse the properties of the graph such as variation in radius and diameter of the graph. We find upper and lower bounds for the new domination number in terms of various already known parameters. Also we obtain several interesting properties like Nordhaus-Gaddum type results relating the graph and its complement.

2. Prior Results

Theorem 2.1:[14]

Let G be any graph and D be any dominating set of G. Then $|V\text{-}D| \leq \sum_{u \in V(D)} \deg(u)$ and equality holds in this relation if and only if D has the following properties :

(i) D is independent.

(ii) For every $u \in$ V-D, There exist a unique vertex $v \in$ D such that $N(u) \cap D = \{v\}$.

Theorem 2.2:[3]

For any tree T of order $p \geq 3$, $\gamma_c(S(T))= 2p-e-1$, where e denotes the number of pendent vertices of T.

Theorem 2.3:[10]

Let be a geodetic graph, which is neither K_2 nor $K_1 \cup K_1$ such that $\overline{G}$ is also geodetic. Then G satisfies one of the following:

(i) G has diameter 3 and radius 2.

(ii) G is self-centered implies $G \cong C_5$.

3. Main Results

3.1. Weak convex dominating sets in Graphs.

Mulder[11] defined interval of a graph G as a subgraph S suchthat <S> includes all shortest paths of G connecting every pair of vertices in S.

Instead of all shortest paths joining every pair of vertices in S, if we consider the inclusion property of at least one shortest of every pair then that will induce a weaker set S of earlier definition of Mulder and hence we define the new concept of weak convex set with domination property as follows.

Definition 3.1 :

A dominating set D with $d_{<D>}(u,v) = d_G(u,v)$ for any two vertices u, v in D is called as a Weak Convex Dominating (W.C.D) set.

The cardinality of a minimum weak convex dominating set of G is called as a weak convex domination number of G and is denoted by γ_{wc}.

Most of the parameters so far defined on domination in graphs is a subclass of this weak convex domination, because the weak convex dominating set is a dominating and distance preserving set of a graph in which it is defined.

Observations :

3.1: Clearly from the definition, $1 \leq \gamma_{wc} \leq p$.

3.2: If G is geodetic, then for any spanning sub graph H, $\gamma_{wc}(G) \leq \gamma_{wc}(H)$.

3.3: For any tree T, $\gamma_{wc}(T)=\gamma_c(T)= p-e$, where e is the number of pendant vertices of T.

3.4: For any graph G $\gamma_{wc}(G^n) \leq \gamma_{wc}(G)$, where G^n is the n^{th} power of a graph G.

3.5: If the diameter of G is n, then $\gamma_{wc}(G^n) = 1$

3.6: Every weak convex dominating set contains a minimal dominating set.

3.7: Every weak convex dominating set of a connected graph contains a minimal connected dominating set.

Clearly, from observations 3.6 and 3.7 we have the relation that $\gamma \leq \gamma_t \leq \gamma_c \leq \gamma_{wc}$. The following Lemma is trivial from the definition of W.C.D set.

Lemma 3.1 :

A weak convex dominating set D is a minimal weak convex dominating set if and only if for each $d \in$ D, one of the following conditions hold:

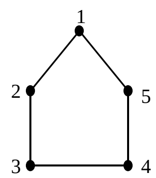
(i) there exists a vertex $c \in$ V-D such that $N(c) \cap D = \{d\}$.

(ii) d must be in the geodesic of two vertices of D satisfying property (i).

(iii) d lies in the geodesic of any two vertices of D, which may satisfy any of the above two properties.

Can V-D be a weak convex dominating set if D is a weak convex dominating set? The answer is that it need not be.

Example 3.1:



Here, D={1, 2, 3} form a W.C.D set, but V-D={4, 5} is not a W.C.D set.

Proposition 3.1 :

Let D be any weak convex dominating set of G. Then $| V\text{-}D | \leq \sum_{u \in V(D)} \deg(u)$, for all $u \in D$.

Proof:

Let D be any weak convex dominating set of G. Then clearly, D is a dominating set of G. Hence from Theorem 2.1, $| V\text{-}D | \leq \sum_{u \in V(D)} \deg(u)$.

Remark 3.2 :

Equality holds for any complete graph.

Proposition 3.2 :

$$| V\text{-}D | = \sum_{u \in V(D)} \deg(u) \iff |D| = 1.$$

Proof:

Since every weak convex dominating set is a dominating set, then the result follows from Theorem 2.1.

Corollary 3.1 :

If $|V\text{-}D| = \sum_{u\in V(D)} \deg(u) \Leftrightarrow$ there exists a vertex of degree p-1.

Lemma 3.2 :

Let G be a connected graph. Then $p\text{-}q \geq 0 \Leftrightarrow$ G is unicyclic or a tree.

Proof :

If G is unicyclic then p-q = 0. If G is a tree then p-q =1. Hence $p\text{-}q \geq 0$. Assume that G is neither unicyclic nor a tree. That is G has at least two cycles. Let H be a spanning tree of G. Then q(H) = p-1. Since G has at least two cycles, $q(G) \geq q(H) + 2$ $\Rightarrow p\text{-}q(G) \leq p\text{-}q(H)\text{-}2 \Rightarrow p\text{-}q \leq -1$. This implies that G is either unicyclic or a tree.

Theorem 3.1 :

Let G be a connected graph. Then $\gamma_{wc}(G) = p\text{-}q \Leftrightarrow$ G is isomorphic to $k_{1,\, r.}$

Proof :

Let $\gamma_{wc} = p\text{-}q>0$. Then by lemma 3.2, G must be a tree, which implies that p-q =1 and hence $\gamma_{wc} = 1$. Therefore radius(G) = 1. Hence $G \cong K_{1,\, r}$.

Proposition 3.3 :

$[\, p/\Delta+1\,] \leq \gamma_{wc}$.

Proof :

Let D be a minimum weak convex dominating set. From proposition 3.1, we have $|\, V\text{-}D\, | \leq \sum_{u\in V(D)} \deg(u) \leq |D|\, \Delta \Rightarrow |V| \leq |D|(\Delta+1) \Rightarrow [\, p/\Delta+1] \leq |D| = \gamma_{wc}$.

Proposition 3.4 :

Let G be a graph of order p. Then $k = \gamma_{wc}(G) = [p/\Delta+1]$ if and only if $\gamma_{wc}(G)=1$.

Proof :

Assume that $k = \gamma_{wc}\,(G) = [\, p/\Delta+1\,]$. Let D be a weak convex dominating set such that $|D| = \gamma_{wc}$.

Claim: D is independent.

If not, let D be a non-independent set. Since D is itself a dominating set and not independent, $|\, V\text{-}D\, | \neq \sum_{u\in V(D)} \deg(u)$.

$$\text{(i.e.)} \quad |\, V\text{-}D\, | < \sum_{u\in V(D)} \deg(u) \leq k\Delta$$

$$\Rightarrow \quad k > p/(\Delta+1) \geq [\, p/\Delta+1\,]$$

This is a contradiction to $k = [\ p/\Delta+1\ ]$. Therefore, D is independent. Since D is a weak convex dominating set, $|D| = 1$, that is $\gamma_{wc}(G) = 1$. Proof of converse is trivial.

Corollary 3.2 :

Let G be a graph of order p such that $\gamma_{wc}(G) = [p/(\Delta+1)]$. Then $\Delta+1$ divides p.

The converse of the above corollary is not true, that is if $\Delta+1$ divides p, then γ_{wc} need not be equal to $[p/(\Delta+1)]$.

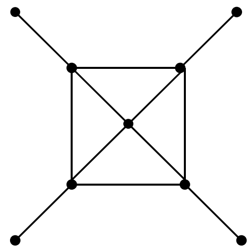
Example 3.2:

Here, $\gamma_{wc} = 7$, but $[p/(\Delta+1)] = 3$.

Remark 3.3:

One can construct a family of infinite number of graphs, which has weak convex dominating sets that have no vertex of G with eccentricity r (radius of G).

Example 3.3:



Theorem 3.2 :

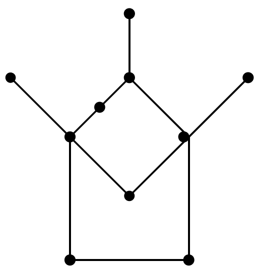
Let G be any distance hereditary graph. Then for any spanning sub graph H of G, $\gamma_{wc}(G) \leq \gamma_{wc}(H)$.

Remark 3.4 :

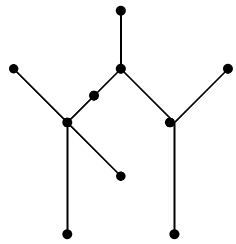
Usually this inequality is not true for any graph G. There are so many graphs, which have the other inequality.

Example 3.4:

G



H



Here $\gamma_{wc}(G) = 5$ and $\gamma_{wc}(H) = 4$

Following question generally arise when one tries to analyse the structural property of W.C.D. set of a graph.

Question:

If d and r are the diameter and radius of the graph G then what about the diameter and radius of the induced graph induced by the convex dominating set of G? The following two theorems give the answer for this question.

Theorem 3.3 :

Let G be a graph and D be a weak convex dominating set. Then the radius of the induced graph <D>, induced by D, is at least r-1, where r is the radius of G.

Proof :

Let *u* be a vertex in the weak convex dominating set D, whose eccentricity is reduced to r-2 in the induced graph <D>. Let *v* be an eccentric point of *u* in G. Then surely *v* does not belong to D (This is because $v \in G$ implies that the shortest path between *u* and *v* must occur in the induced graph and it is greater than are equal to r. This contradicts our assumption). Then v must be dominated by some vertex say, *w* in D. Clearly, $d(u, w) \leq$ r-2 in the induced graph <D>.

This implies that $d(u, w) + d(w, v) \leq$ r-1, that is $d(u, v) \leq$ r-1 in G. This contradicts the fact that the distance between *u* and *v* is greater than or equal to r. Hence, there is no point in the weak convex dominating set, which has eccentricity r-2 in the induced graph <D>.

Remark 3.5:

It is easy to verify that the radius of the induced graph <D> cannot be increased more than the diameter of the original graph.

Corollary 3.3:

Let d and r be the diameter and radius of the given graph G and let r_c be the radius of the induced graph induced by the weak convex dominating set of G. Then r-1 $\leq r_c \leq$ d.

The following two theorems gives the picture of diameter of W.C.D. set of a graph.

Theorem 3.4:

There exists no graph, which has a weak convex dominating set that induces a sub graph of diameter less than are equal to d-3, where d is the diameter of the original graph.

Proof :

Let d be the diameter of the given graph. To get an induced sub graph of diameter d-3, we have to eliminate at least three edges from all diameteral paths. We can eliminate three edges from the diameteral path only by the following two ways :

(i) Two edges from one end and one edge from the other end.

or (ii) three consecutive edges from one end.

But in both the cases domination property is lost. Hence there exists no graph, which has a weak convex dominating set that induces a sub graph of diameter less than or equal to d-3.

Theorem 3.5 :

Let d, d_c be the diameters of the given graph G and the induced sub graph of the weak convex dominating set of G respectively. Then d-2 $\leq d_c \leq$ d.

Proof :

Proof follows from the fact of Theorem 3.4.

Corollary 3.3:

Let G be a connected graph with diameter d. Then d-1 $\leq \gamma_{wc}(G)$.

Proof :

Proof follows from Theorem 3.5.

Following propositions are direct and hence we state without details of proofs.

Proposition 3.5 :

Let cut(G) denote the number of cut vertices of a connected graph G. Then cut(G) $\leq \gamma_{wc}(G)$.

Proposition 3.6 :

Let G be a self-centered graph of diameter 2. Then $\gamma_{wc}(G) \leq \delta+1$.

Proof:

Any δ degree vertex with its first neighborhood forms a W.C.D. set and hence the proposition.

Proposition 3.7 :

Let G be a graph with radius 2. Then $\gamma_{wc}(G) \leq \Delta+1$.

Proposition 3.8 :

If $\gamma_{wc}(\bar{G}) \geq 3$, then diam(G) ≤ 3.

Theorem 3.6 :

If a graph G is connected and diam(G) ≥ 3, then $\gamma_{wc}(\bar{G}) = 2$.

Theorem 3.7 :

If a graph G has $\delta \geq 2$ and girth g(G) ≥ 7, then $\gamma_{wc}(G) = p$.

Proof :

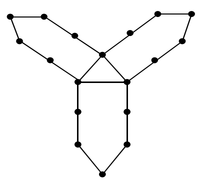
Let D be a γ_{wc} set of G. We know that $\gamma_{wc}(G) \leq p$. If $\gamma_{wc}(G) \neq p$, then $\gamma_{wc}(G) < p$. This implies that $|V\text{-}D| \geq 1$. Let $u \in V\text{-}D$. If two vertices of D dominate u, then as D is convex there must exists a C_3 or C_4 in G. Therefore, only one vertex of D dominates u. Since $\delta(G) \geq 2$, there must exists another vertex $v \in V\text{-}D$ such that u and v are adjacent. And also u and v are not be dominated by the same vertex of D (if possible, then C_3 arises). Therefore, u and v are dominated by two different vertices say some u' and v' of D respectively. This implies that $d(u', v') \leq 3$ (since length of the path u'uvv' = 3). This implies that there must be a C_4 or C_5 or C_6 exist in G, which is a contradiction to g(G) $\geq$ 7. Hence $\gamma_{wc}(G) = p$.

Remark 3.6:

Is converse of the above true?

Need not be. Consider the following :

Example 3.5:



For the above graph g(G) = 3 and $\gamma_{wc}(G) = p$.

Theorem 3.8:

For any connected graph G such that $\bar{G}$ also connected , $\gamma_{wc}(G) + \gamma_{wc}(\bar{G}) \leq p+2$, where $\gamma_{wc}(G)$ and $\gamma_{wc}(\bar{G})$ are the cardinality of minimum weak convex sets of G and $\bar{G}$ respectively.

Proof :

Case 1: For r = 1, d = 1 and r = 1, d = 2.

Clearly, there is no graph G such that G and $\bar{G}$ are connected.

Case 2 : For r = 2 , d = 2.

Consider a vertex *v*. Clearly, $\{v\} \cup N_1(v)$ forms a weak convex dominating set for G . We have the property that every vertex in $N_1(v)$ has at least one eccentric point in $N_2(v)$. That is every point in $N_1(v)$ is not adjacent to at least one point of $N_2(v)$. Also every point of $N_2(v)$ is not adjacent to v. Hence $\{v\} \cup N_2(v)$ forms a weak convex dominating set for $\bar{G}$. Hence, $\gamma_{wc}(G)+\gamma_{wc}(\bar{G}) \leq p+1$.

Case 3 : For $r \geq 2$, $d \geq 3$.

In G, there must exist at least two vertices with distance between them is greater than or equal to 3. Then that two vertices form a weak convex dominating set for $\bar{G}$. Hence, $\gamma_{wc}(G) + \gamma_{wc}(\bar{G}) \leq p+2$.

Remark 3.7:

The cycles of order greater than or equal to 7 attain this bound p+2.

Theorem 3.9:

For a given graph G,

$$\gamma_{wc}(G) \cdot \gamma_{wc}(\bar{G}) \begin{cases} \leq p & ; \text{ if } r=1 \text{ or } \bar{r}=1 \\ \leq (\delta+1)^2 & ; \text{ if } d=\bar{d}=2 \\ \leq 2p & ; \text{ if } d \text{ or } \bar{d} \geq 3, \end{cases}$$

where (r, d) and $(\bar{r}, \bar{d})$ denote the radius and diameter of G and $\bar{G}$ respectively.

Definition 3.2:

A graph is said to be k-weak convex dominating special graph, if it has exactly k-disjoint weak convex dominating sets.

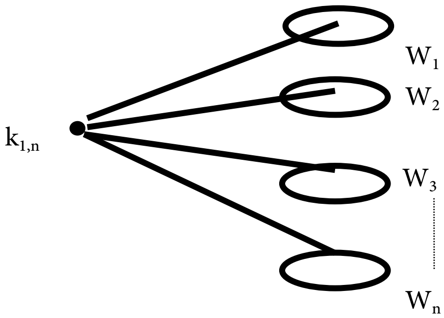
If the whole vertex set of G is the only W.C.D set of G, then G is said to be a 0-weak convex dominating graph.

Remark 3.8:

Whether all 0-weak convex dominating graphs are 2-connected ?

The answer is no, we could construct a family of graphs by attaching several 0-weak convex dominating graphs at pendent vertices of $K_{1,n}$ for all positive integer n.

Example 3.6 :



Separable 0-weak convex graphs.

Here W_1, W_2, …, W_n are 0-weak convex graphs.

Proposition 3.9 :

A graph G is 0-weak convex dominating graph, then the diameter d of G is greater than or equal to 3. The smallest graph is C_7.

Proof :

Follows from the fact that if G has diameter less than or equal to 2 then $\gamma_{wc} \leq \Delta+1$ and hence a contradiction to G is 0-weak convex dominating graph.

Proposition 3.10:

A graph G is 0-weak convex dominating graph, then $\bar{G}$ is not 0-weak convex dominating graph.

Proof:

As the graph has diameter greater than or equal to 3, $\bar{G}$ has a dominating edge and hence $\gamma_{wc}(\bar{G})=2$. Hence the proposition.

Following corollary is immediate from the previous two propositions.

Corollary 3.4 :

There is no graph G such that both G and $\bar{G}$ are 0-weak convex dominating graphs.

Theorem 3.10:

For any graph G , $\gamma_{wc}(G) + \gamma_{wc}(\bar{G}) = p+2 \Leftrightarrow$ one of the graphs G or $\bar{G}$ is a 0-weak convex dominating graph.

Proof :

Let G be any graph. Assume that $\gamma_{wc}(G) + \gamma_{wc}(\bar{G}) = p+2$. Then clearly, $\gamma_{wc}(G)$ and $\gamma_{wc}(\bar{G})$ are not less than are equal to 2 (from the fact of proof of theorem 3.8). Thus, either diam(G) or diam($\bar{G}$) is greater than or equal to 3. Without loss of generality assume that diam(G) is greater than or equal to 3. Then clearly, $\gamma_{wc}(\bar{G}) = 2$ and hence, $\gamma_{wc}(G)+\gamma_{wc}(\bar{G}) = p+2$ implies that $\gamma_{wc}(G) = p$, that is G is 0-weak convex dominating graph.

Conversely, assume that G or $\bar{G}$ is a 0-weak convex dominating graph. Without loss of generality, let G be a 0-weak convex dominating graph, that is $\gamma_{wc}(G) = p$. Then from the proposition 3.9, diam(G) $\geq$ 3. This implies that $\gamma_{wc}(\bar{G}) = 2$. Hence, $\gamma_{wc}(G) + \gamma_{wc}(\bar{G}) = p+2$.

The following proposition gives the relation between p, q and γ_{wc}.

Proposition3.11:

If G is a (p, q) - graph, then $q \geq \frac{1}{2}(p + \gamma_{wc})$.

Proof:

Clearly from the previous proposition, every pendent vertex of G belongs to V – D. This implies that deg(u) $\geq$ 2 for all u $\in$D.

Therefore, $2q = \sum_{u\in V(G)} \deg(u) = 2\,|D| + |V - D| = 2\gamma_{wc} + p - \gamma_{wc} = \gamma_{wc} + p$.

Hence, $q \geq \frac{1}{2}(p + \gamma_{wc})$.

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A New Form of Generalized Yield Criteria of Porous Sintered Powder Metallurgy Metals

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Abstract: *This note is intended to present a new and corrected version of generalized yield criteria of porous sintered powder metallurgy metals, which was proposed in our earlier paper (Narayanasamy et.al.,2001). It is noticed that equations (14)-(17) are dimensionally and mathematically incorrect. This is a slip in our part. In the case of anisotropic non-porous sheet metals with the parameters m=2, n=2 and p=1, mathematical expression for yield equation obtained from generalized yield equation (see equation (15)) in our paper (Narayanasamy et.al.,2001) has not been reduced to the corresponding cited equation (20) which was taken from the book written by Johnson and Mellor(1973). It is wrongly mentioned in our paper (Narayanasamy et.al., 2001) that when substituting m=2, p=1 and n=2 in Eq.(17), Eq.(24) is obtained, which is impossible. It is further observed that Eqs.(14) and (22) are not the same when m=2 and p=1. At this juncture, it is pertinent to pin-point out that the theoretical prediction of forming limit strain, wrinkling tendency, localized necking behaviour etc. of sheet materials using improper generalized yield criteria leads to wrong conclusions which, in turn, could not be helpful for us to have a crystal clear understanding the actual characteristics of plastic deformation of sheet metals and materials and to the development of designing tools for manufacturing process. Based on the forgoing views, it is pivotal to propose a new form of genreralized yield equation and obtain the corresponding particular cases that form the content of the present paper.*

Key words: *New and corrected version of Generalized yield equation; porous-anisotropic metals; P/M perform*

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Case 1: Generalized Yield Theory

As reported by Lee and Kim (1992), we write

$$(2+R^n)J_2^{'}+\frac{1}{3}(1-R^n)J_1^2=Y_R^2 \tag{1}$$

where J_1, $J_2^{'}$ are called linear stress invariant and the quadratic stress deviator invariant respectively. It is observed from the work done by Lee and Kim (1992) that the value of n has been taken as 2 where n is a constant, which takes values like 1.80,1.9,2.0 and 2.1, depending upon the level of anisotropic nature. The yield stress of porous metal Y_R may be related to the yield stress of the non-porous metal Y_0 through a geometrical hardening parameter (η) as follows.

$$Y_R^2=\eta Y_0^2 \tag{2}$$

where $\eta = \{(R\text{-}R_c) / (1\text{-}R_c)\}^2$ with $0 < R_c \le R \le 1$,

which is given in Lee and Kim (1992). Here R and R_c are relative density and critical relative density of P/M perform respectively. Yield equation for anisotropic metals in terms of the principal stresses can obtained as follows.

$$Y_0^m(1+r^p)=\left|\sigma_2-\sigma_3\right|^m+\left|\sigma_3-\sigma_1\right|^m+r^p\left|\sigma_1-\sigma_2\right|^m \tag{3}$$

where p is a constant r^p varies from say, 0.8 to 1.2. The volumetric strain energy expression can be written as:

$$J_1^m=(\frac{3}{2+r^p})\left|\sigma_1+\sigma_2+\sigma_3\right|^m \tag{4}$$

The modified distortion strain energy term is

$$J_2' = \left\{ \frac{1}{2+r^p} \right\} \frac{\{|\sigma_2 - \sigma_3|^m + |\sigma_3 - \sigma_1|^m + r^p |\sigma_1 - \sigma_2|^m\}^{\frac{2}{m}}}{(2+(p-1)^2 r^p)} \tag{5}$$

From Eqs.(1)-(5), the new and corrected version of yield criterion with five parameter constants for porous anisotropic metals becomes;

$$\begin{aligned} &\frac{(2+R^n)}{(2+r^p)(2+(p-1)^2 r^p)} \{|\sigma_2 - \sigma_3|^m + |\sigma_3 - \sigma_1|^m + r^p |\sigma_1 - \sigma_2|^m\}^{\frac{2}{m}} \\ &+ \left(\frac{3}{2+r^p} \right)^{\frac{2}{m}} \frac{(1-R^n)}{3} |\sigma_1 + \sigma_2 + \sigma_3|^2 = Y_0^2 \frac{(R-R_c^2)}{(1-R_c)^2} \end{aligned} \tag{6}$$

For Uniaxial compression or porous –anisotropic metals, Eq.(6) becomes:

$$\frac{(1+r^p)^{\frac{2}{m}}}{(2+(p-1)^2 r^p)} = \frac{(2+R^n)}{2+r^p} \sigma_1^2 + \left(\frac{3}{2+r^p} \right)^{\frac{2}{m}} \frac{(1-R^n)}{3} \sigma_1^2 = Y_R^2$$

(7)

Eq.(7) can be rewritten as follows:

$$\frac{\sigma_1^2}{Y_0^2} \left[\frac{(1+r^p)^{\frac{2}{m}}}{(2+(p-1)^2 r^p)} \frac{(2+R^n)}{(2+r^p)} + \frac{\left(\frac{3}{2+r^p} \right)^{\frac{2}{m}} (1-R^n)}{3} \right] = \frac{(R-R_c^2)}{(1-R_c)^2} \tag{8}$$

Case 2: Porous Anisotropic Metals

The yield stress of a porous metal Y_R may be related to the yield stress of a non-porous metal Y_0 through a geometrical hardening parameter (η) as follows, as given in case 1.

$$Y_R^2 = \eta Y_0^2 \tag{9}$$

where $\eta = \{(R-R_c) / (1-R_c)\}^2$ with $0 < R_c \leq R \leq 1$,

which is reported by Lee and Kim(1992). Here R and R_c are relative density and critical relative density of P/M perform respectively. Yield equation for anisotropic metals in terms of the principal stresses can obtained as follows(Johnson and Mellor,1973):

$$\bar{\sigma} = \sqrt{\frac{3}{2}}[\frac{(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2+r(\sigma_1-\sigma_2)^2}{(2+r)}]^{\frac{1}{2}} \tag{10}$$

Substituting the values of n = m = 2 and p =1 in Eq.(6), the yield criterion for porous-anisotropic metals becomes:

$$(2+R^2)\frac{1}{2}[\frac{(\sigma_1-\sigma_2)^2 r+(\sigma_2-\sigma_3)^2+(\sigma_2-\sigma_3)^2}{(2+r)}] + \left(\frac{1-R^2}{3}\right)[\frac{3}{(2+r)}(\sigma_1+\sigma_2+\sigma_3) = Y_0^2\frac{(R-R_c^2)}{(1-R_c)^2} \tag{11}$$

For Uni-axial compression of porous-anisotropic metals, Eq.(11) becomes:

$$\frac{\sigma_1^2}{Y_0^2} = \frac{(R-R_c^2)(4+2r)}{[(2+R^2)(1+r)+2(1-R^2)](1-R_c)^2} \tag{12}$$

When substituting m = n = 2 and p = 1 in Eq.(8) , Eq.(12) is obtained.

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Domination Parameters Of Hypercubes

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Abstract: *Let G be a connected graph. Let γ, γ_i, γ_t, γ_c and γ_p denote respectively the domination number, the independent domination number, the total domination number, the connected domination number and the perfect domination number of G. The n-cube Q_n is the graph, whose vertex set is the set of all n-dimensional Boolean vectors with two vertices being joined if and only if they differ in exactly one coordinate. Hamming proved that Q_n has a perfect dominating set if and only if $n = 2^k-1$. Here, it is proved that $\gamma(Q_n) = 2^{n-k}$ for $n = 2^k$. Bounds for $\gamma_i(Q_n)$, $\gamma_c(Q_n)$, $\gamma_t(Q_n)$ and $\gamma_p(Q_n)$ are also found out. Finally, it is conjectured that $\gamma(Q_n) = \gamma(Q_{n-1}) + \lceil (2^{n-1}-\gamma(Q_{n-1}))/(n-1) \rceil$ for $2^k+1 \le n \le 2^{k+1}-2$, where $\lceil x \rceil$ denote the least integer not less than x.*

1. Introduction

Let G be a finite, connected, undirected, simple graph with vertex set V(G) and edge set E(G). A vertex u is said to dominate the vertex v if E(G) contains an edge from u to v or if u = v. A set D $\subseteq$ V(G) is a *dominating set*, if every vertex in V(G) is either an element of D or is adjacent to an element of D; that is every vertex of G is dominated by at least one member of D. A dominating set D is an *independent dominating* set, if no two vertices in <D> are adjacent, that is, D is an independent set. A dominating set D is a *connected dominating* set, if < D > is a connected subgraph of G. A dominating set D is a *perfect dominating* set, if each vertex of G is dominated by exactly one element of D. Clearly every perfect dominating set is independent dominating set. A dominating set D is a *total dominating* set, if < D > has no isolated vertex. The domination number γ of G is defined to be the minimum cardinality of a dominating set in G. Similarly, we can define the perfect domination number γ_p, connected domination number γ_c, total domination number γ_t, independent domination number γ_i for a graph G. It is clear that a perfect dominating set for a graph is necessarily a minimum dominating set. A dominating set with cardinality γ(G) is known as a γ*-dominating set*. The *domatic number* d(G) of a graph G is the maximum number of elements in a partition of V(G) into dominating sets. The *distance* d(u, v) between two vertices u and v in G is the minimum length of a path joining them. Let D_1 and D_2 be two subsets of V(G). Distance between the two sets D_1, D_2

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is defined as the minimum of $\{d(u, v) : u \in D_1, v \in D_2\}$. The definitions and details not furnished here may be found in [2] and [5]. The hypercube or n-cube Q_n is the graph whose vertex set is the set of all n-dimensional Boolean vectors in which two vertices are joined if and only if they differ in exactly one coordinate. We observe that $Q_1 = K_2$ and $Q_n = Q_{n-1} \times K_2$ if $n \geq 2$.

A communication network can be represented by a connected graph G, where the vertices of G represent processors and edges represent bi-directional communication channels. The hypercube Q_n of dimension n is one of the most versatile and powerful interconnection networks. It has been successfully employed in the architecture of massively parallel computers. A dominating set in Q_n can be interpreted as a set of processors from which information can be passed on to all the other processors. Hence, the determination of the domination parameters of Q_n is a significant problem.

The notion of perfect dominating set in a hypercube is same as that of a single-error correcting binary code, which is due to R.W.Hamming[4]. Jha [6] has proved the following theorem:

Theorem 1.1[6] Let $\gamma(Q_n)$ denote the domination number of Q_n. Then $[2^n/(n+1)] \leq \gamma(Q_n) \leq [2^n/2^h]$, where $h = \lfloor \log_2(n+1) \rfloor$. If $n = 2^k - 1$, then the two bounds coincide and hence $\gamma(Q_n) = 2^{n-k}$. These bounds are correct also for $\gamma_i(Q_n)$.

R.W.Hamming [4] proved the following theorem,

Theorem 1.2 [4] Q_n has a perfect single-error correcting code if and only if $n = 2^k - 1$. Arumugam et al [1] have independently proved the same theorem.

Zelinka [8] has proved the following theorem:

Theorem 1.3[8] Let k be a positive integer. Then the graph of the cube of dimension $2^k - 1$ and the graph of the cube of dimension 2^k have both the domatic number 2^k.

Jaeun Lee has proved the following:

Theorem 1.4[7] Let n be a natural number. Then the following are equivalent. (1) The hypercube Q_n has an independent perfect dominating set. (2) $n = 2^m - 1$ for a natural number m.

Here, we independently prove that $\gamma(Q_n) = 2^{n-k}$ when $n = 2^k$, and we obtain bounds for γ, γ_i, γ_t, and γ_c whenever $2^k \leq n \leq 2^{k+1} - 2$. We also conjecture that, $\gamma(Q_n) =$

$\gamma(Q_{n-1}) + \lceil (2^{n-1}-\gamma(Q_{n-1}))/(n-1) \rceil$ for $2^k+1 \leq n \leq 2^{k+1}-2$, where $\lceil x \rceil$ denote the least integer not less than x.

For brevity, take $m = 2^k$ throughout this paper. Thus $2m = 2^{k+1}$

2.Structure of the Hypercubes Q_n

The n-cube Q_n is the graph whose vertex set is the set of all n-dimensional Boolean vectors; two vectors being joined if and only if they differ in exactly one coordinate. We observe that $Q_1 = K_2$, $Q_n = Q_{n-1} \times K_2$, if $n \geq 2$. Now, Q_n can be viewed as follows:

Let $t = 2^{n-1}$ and let $(Q_{n-1})^0$, $(Q_{n-1})^1$ be two copies of Q_{n-1} with vertex sets $V((Q_{n-1})^0) = \{u_1^0, u_2^0,\ldots, u_t^0\}$, $V((Q_{n-1})^1) = \{u_1^1, u_2^1,\ldots, u_t^1\}$, where $u_i \in Q_{n-1}$.

Then $V(Q_n) = V((Q_{n-1})^0) \cup V((Q_{n-1})^1)$,

$E(Q_n) = E((Q_{n-1})^0) \cup E((Q_{n-1})^1) \cup \{ u_i^0u_i^1 : i = 1, 2,\ldots, t\}$. Similarly,

$Q_{n+1} = Q_n \times K_2 = (Q_{n-1} \times K_2) \times K_2$. Let $(Q_{n-1})^{00}$, $(Q_{n-1})^{10}$, $(Q_{n-1})^{01}$, $(Q_{n-1})^{11}$ be 2^2 copies of Q_{n-1} with $V((Q_{n-1})^{00}) = \{ u_1^{00}, u_2^{00},\ldots, u_t^{00} \}$, $V((Q_{n-1})^{10}) = \{ u_1^{10}, u_2^{10},\ldots, u_t^{10} \}$, $V((Q_{n-1})^{01}) = \{ u_1^{01}, u_2^{01},\ldots, u_t^{01} \}$, $V((Q_{n-1})^{11}) = \{ u_1^{11}, u_2^{11},\ldots, u_t^{11} \}$, and

$$V(Q_{n+1}) = V((Q_{n-1})^{00}) \cup V((Q_{n-1})^{01}) \cup V((Q_{n-1})^{10}) \cup V((Q_{n-1})^{11}),$$

$$\begin{aligned} E(Q_{n+1}) = {} & E((Q_{n-1})^{00}) \cup E((Q_{n-1})^{10}) \cup E((Q_{n-1})^{01}) \cup E((Q_{n-1})^{11}) \\ & \cup \{ u_i^{00}u_i^{01} : u_i \in Q_{n-1}\} \cup \{ u_i^{00}u_i^{10} : u_i \in Q_{n-1}\} \\ & \cup \{ u_i^{10}u_i^{11} : u_i \in Q_{n-1}\} \cup \{ u_i^{01}u_i^{11} : u_i \in Q_{n-1}\}. \end{aligned}$$

That is, edges of Q_{n+1}'s are just the union of edges of $(Q_{n-1})^{00}$, $(Q_{n-1})^{10}$, $(Q_{n-1})^{01}$, $(Q_{n-1})^{11}$ and the edges joining u_i^{xy} and u_i^{pq}, where (x,y) , (p,q) are two dimensional Boolean vectors differing exactly in one place. Let $(Q_{n-1})^{000}$, $(Q_{n-1})^{100}$, $(Q_{n-1})^{010}$, $(Q_{n-1})^{110}$, $(Q_{n-1})^{001}$, $(Q_{n-1})^{101}$, $(Q_{n-1})^{011}$, $(Q_{n-1})^{111}$ be 2^3 copies of Q_{n-1} and $V((Q_{n-1})^{xyz}) = \{u_1^{xyz}, u_2^{xyz}, \ldots, u_t^{xyz}\}$. Then $V(Q_{n+2}) = \cup V((Q_{n-1})^{xyz})$, $E(Q_{n+2}) = \cup E((Q_{n-1})^{xyz}) \cup \{u_i^{xyz}u_i^{pqr} : (x,y,z), (p,q,r)$ denote Boolean vectors differing at exactly one place$\}$. Similarly, we can view Q_{n+3} ,…etc, in terms of Q_{n-1} or Q_{n+3}, Q_{n+4}… etc in terms of Q_n etc.

Q_n consists of 2^k pairwise vertex-disjoint copies of Q_{n-k}.

For each binary k-tuple x in Q_k and for each binary (n-k)-tuple y in Q_{n-k}, let f(x, y) = xy, the concatenation of x and y. Clearly f is a 1-1 correspondance between the Cartesian product $Q_k \times Q_{n-k}$ and Q_n, and it is easy to see that f is in fact edge preserving and therefore a graph isomorphism. Define $(Q_{n-k})^x$ to be $\{xy \mid y \in Q_{n-k}\}$. The family $\{(Q_{n-k})^x \mid x \in Q_k\}$ gives the desired vertex partition of Q_n into 2^k copies of Q_{n-k}.

Proposition 2.1 Let S be any collection of k-dimensional Boolean vectors with $k \geq 4$ such that any two of them differ in exactly two places. Then S contains exactly k elements.

Proof: .Let x be any vertex of Q_k, and let S_1 be the set of all neighbors of x. Then $|S_1| = k$, and if $y,z \in S_1$, then $d(y, z) = 2$ (since y,x,z is a path). Therefore, S_1 is a collection of k-dimensional Boolean vectors such that any two of them differ in exactly two places. Hence $|S| \geq k$ —— (1)

Suppose S contains more than k elements, S contains at least k+1 elements x_1, x_2, …, x_{k+1}.

Claim: There exists at least one pair x_i, x_j, i, j >1 , $i \neq j$ such that x_i and x_j will not differ in exactly two places.

Proof of the claim:

Think of the vertices of Q_k as the subset of [k] = {1, 2, 3, …, k}. By the vertex symmetry of Q_k, we may assume that $x_{k+1} = 0$. Then for $1 \leq i \leq k$, x_i is a 2-subset of [k]. By assumption, x_i and x_j differ in exactly 2 places, i.e. $|x_i \Delta x_j| = 2$. Now $|x_i \Delta x_j| = |x_i| + |x_j| - 2|x_i \cap x_j| = 2 + 2 - 2|x_i \cap x_j|$. Thus $|x_i \cap x_j| = 1$. Without loss of generality, we may assume that $x_1 = \{1, 2\}$. We claim: either $1 \in x_i$ for all $2 \leq i \leq k$ or $2 \in x_j$ for all $2 \leq j \leq k$. For if not, there exists an i such that $x_i = \{2, a\}$ and there exists a j such that $x_j = \{1, b\}$, with a, b ≥ 3. Since $|x_i \cap x_j| = 1$, a = b. So $x_j = \{1, 3\}$ and $x_i = \{2, 3\}$. Since $k \geq 4$, there exists an x_l with $l \neq 1, i, j$.

Case 1: $1 \in x_l$. Hence $2 \notin xl$ (or else xl = x1). Then since $|x_i \cap x_l| = 1$, $x_i \cap x_j = \{1\}$ and $3 \notin x_l$. Then $x_i \cap x_j = \phi$, a contradiction.

Case 2: $2 \in x_l$. Hence $1 \notin x_l$ Also, $3 \notin x_l$ since otherwise $x_l = x_i$. Then since $|x_i \cap x_l| = 1$, $x_l \cap x_j = \phi$, again a contradiction.

This proves our claim. Without loss of generality, assume $1 \in x_i$ for all i = 1, 2, …, k. But there are only k−1 2-subsets of [k] which contain 1 as a member. This contradiction proves that S cannot have k+1 elements. Hence, $|S| \leq k$.

From (1) and (2), we see that $|S| = k$..

Remark 2.1 We know $Q_n = Q_{n-1} \times K_2$. $V(Q_n) = V((Q_{n-1})^0) \cup V((Q_{n-1})^1)$ and $E(Q_n) = E((Q_{n-1})^0) \cup E((Q_{n-1})^1) \cup \{u_i^0u_i^1: i = 1, 2, 3, ..., 2^{n-1}, u_i^0 \in V((Q_{n-1})^0) , u_i^1 \in V((Q_{n-1})^1)\}$. Let D be a dominating set of Q_{n-1}. Let D^0, D^1 be the corresponding dominating sets of $(Q_{n-1})^0$, $(Q_{n-1})^1$. Then (i) $D^0 \cup D^1$ is a total dominating set for Q_n,

(ii) $D^0 \cup \{V((Q_{n-1})^1) - D^1\}$ is a dominating set for Q_n. Also if $S^1 \subseteq V((Q_{n-1})^1)$, then $D^0 \cup S^1$ is a dominating set for Q_n if and only if S^1 dominates $< V((Q_{n-1})^1) - D^1 >$.

Proposition 2.2 If $n = 2^k-1$, vertices of Q_n can be partitioned into 2^k (= m) perfect dominating sets $D_1, D_2, D_3, ..., D_m$ each containing exactly 2^{m-k-1} elements.

Proof: For if D_1 is the perfect single-error correcting Hamming code on Q_n, D_1 is linear, i.e. a subgroup of Q_n under vector addition. We take $D_2, D_3, \ldots, D_m$ to be the other cosets of D_1 in Q_n, thereby giving a partition of Q_n into perfect dominating sets, all translates of D_1 under addition. Thus all are perfect dominating sets, and have the same size.

Remark 2.2 Consider $Q_{m+1} = (Q_{m-1} \times Q_2)$. It contains four disjoint copies of Q_{m-1} as subgraphs: $(Q_{m-1})^{00}$, $(Q_{m-1})^{01}$, $(Q_{m-1})^{10}$, $(Q_{m-1})^{11}$. We know $\gamma(Q_{m-1}) = \gamma_i(Q_{m-1}) = \gamma_p(Q_{m-1})$. Let D_1, D_2 be any two disjoint perfect dominating sets of $Q_{m-1,}$ which exist since by the standing hypothesis $m = 2^k$. Let $(D_1)^{00}$, $(D_2)^{11}$ be the corresponding dominating sets of $(Q_{m-1})^{00}$, $(Q_{m-1})^{11}$ respectively. Clearly, distance between $(Q_{m-1})^{00}$, $(Q_{m-1})^{11}$ is two, and distance between $(D_1)^{00}$, $(D_2)^{11}$ is three. Therefore, $(D_1)^{00} \cup (D_2)^{11}$ is an independent, perfect subset of $V(Q_{m+1})$. Similarly, $(D_1)^{01} \cup (D_2)^{10}$ is an independent perfect subset of $V(Q_{m+1})$. Note that neither of these is a dominating set since $m+1 = 2^k+1 \neq 2^q - 1$ for any q.

3.Domination of Q_n when $n = 2^k$

Theorem 3. 1 If $m = 2^k$, then $\gamma(Q_m) = 2^{m-k}$.

Proof: We have $Q_m = Q_{m-1} \times K_2$

Let $Q_{m-1}{}^0$, $Q_{m-1}{}^1$ be two copies of Q_{m-1}. Then $V(Q_m) = V(Q_{m-1}{}^0) \cup V(Q_{m-1}{}^1)$ and $E(Q_m) = E(Q_{m-1}{}^0) \cup E(Q_{m-1}{}^1) \cup \{u_i^0 u_i^1 : u_i \in V(Q_{m-1}) \text{ and } i = 1, 2, \ldots, t\}$ where $t = 2^{m-1}$

We know that $\gamma(Q_{m-1}) = \gamma_p(Q_{m-1}) = 2^{m-k-1}$.

Therefore, $\gamma(Q_m) \leq 2\gamma(Q_{m-1}) = 2^{m-k}$. ------- ------------------- (I)

Claim: $\gamma(Q_m) \geq 2^{m-k}$.

Let $D = S^0 \cup S^1$ be a minimum dominating set of Q_m where, $S^0 \subseteq V(Q_{m-1}{}^0)$ and $S^1 \subseteq V(Q_{m-1}{}^1)$.

Case 1: S^0 is a minimum dominating set of $Q_{m-1}{}^0$.

$|S^0| = 2^{m-k-1}$ and distance between any two vertices of S^0 is greater than or equal to 3. S^0 dominate all the vertices of $Q_{m-1}{}^0$ and 2^{m-k-1} vertices of $Q_{m-1}{}^1$. Hence S^0 dominates $2^{m-1} + 2^{m-k-1}$ vertices of Q_m.

Let $S^0 = \{v_i^0 / \ i = 1, 2, \ldots, 2^{m-k-1}, v_i \in V(Q_{m-1}) \text{ and } d(v_i, v_j) \geq 3\}$. Let $S = \{v_i^1 / \ i = 1,2,\ldots 2^{m-k-1}, v_i^0 \in S^0\} \subseteq V(Q_{m-1}{}^1)$. Then S^0 dominates $Q_{m-1}{}^0$ and S. $V(Q_{m-1}{}^1) - S$ contains $2^{m-1} - 2^{m-k-1}$ vertices each one is of degree $m-2$ in $V(Q_{m-1}{}^1) - S$. Hence minimum number of vertices needed to dominate these vertices are $(2^{m-1} - 2^{m-k-1})/(m-1) = (2^{m-k-1}(2^k-1))/(m-1)$

$= 2^{m-k-1}$, since $m = 2^k$.

Hence $|S^1| \geq 2^{m-k-1}$. So, $|D| = |S^0| + |S^1| \geq 2.\ 2^{m-k-1} = 2^{m-k}$

Hence $\gamma(Q_m) \geq 2^{m-k}$

Note: If S^1 is a minimum dominating set of Q_{m-1}^1 then we can prove $|S^0| \geq 2^{m-k-1}$.

Case 2(a): S^0, S^1 are not minimum dominating sets of Q_{m-1}^0, Q_{m-1}^1 respectively , but $|S^0| = 2^{m-k-1}$.

S^0 is not a minimum dominating set of Q_{m-1}^0 and hence it is not a dominating set of Q_{m-1}^0. That is S^0 is not dominating Q_{m-1}^0 and $|S^0| = 2^{m-k-1}$. This implies that there exist at least $(m-2)$ elements of Q_{m-1}^0 which are not dominated by S^0.

To dominate these vertices in Q_m, $(m-2)$ vertices must be included in S^1. These $(m-2)$ vertices dominate at most $(m-1)(m-2)$ other vertices in Q_{m-1}^1. Hence remaining vertices in $Q_{m-1}^1 = 2^{m-1} - (m-2) - (m-1)(m-2) = 2^{m-1} - (m-2)m$

To dominate these vertices at least $(2^{m-1} - (m(m-2)))/m$ vertices must be included in S^1.

Hence $|S^1| \geq (m-2) + (2^{m-1} - (m(m-2)))/m$

$= (2^{m-1})/m = 2^{m-k-1}$

Hence $|D| = |S^0| + |S^1| \geq 2.\ 2^{m-k-1} = 2^{m-k}$

$\gamma(Q_m) \geq 2^{m-k}$

<u>Case 2(b):</u> S^0, S^1 are not minimum dominating sets of Q_{m-1}^0, Q_{m-1}^1 respectively and either $|S^0| < 2^{m-k-1}$ or $|S^1| < 2^{m-k-1}$.

Let us assume that $|S^0| < 2^{m-k-1}$.

<u>Subcase 1:</u> $|S^0| = 2^{m-k-1} - 1$.

Since $\gamma(Q_{m-1}) = \gamma_p(Q_{m-1}) = 2^{m-k-1}$, there exist at least one $u^0 \in V(Q_{m-1}^0)$ such that elements of $N[u^0]$ is not dominated by any vertex of S^0 and $|N[u^0]| = m$.

So to dominate these vertices of $N[u^0]$ in Q_m, m vertices of $V(Q_{m-1}^1)$ must be included in D and hence in S^1.

These vertices are nothing but u^1 and neighbours of u^1 in Q_{m-1}^1.

These m elements dominate at most $(m-1)(m-2)$ vertices of Q_{m-1}^1 other than elements of $N[u^1]$. (u^1dominate $(m-1)$ elements which are elements of $N[u^1]$ only and an element $v \in N[u^1]$, $v \neq u^1$ dominate $(m-2)$ elements which are not in $N[u^1]$).

So, remaining vertices of Q_{m-1}^1 is $2^{m-1} - m - (m-1)(m-2)$

$= 2^{m-1} - m - m^2 + 3m - 2$

$= 2^{m-1} - m^2 + 2m - 2.$

To dominate these vertices, at least $(2^{m-1} - m^2 + 2m - 2)/m$ vertices of Q_{m-1}^1 are needed.

Therefore, $|S^1| \geq m + (2^{m-1} - m^2 + 2m - 2)/m$

$= (2^{m-1})/m + 2 - 2/m$

$= (2^{m-1})/m + 1 + (1 - 2/m)$

$\geq 2^{m-k-1}+1.$

Hence, $|D| = |S^0| + |S^1| \geq (2^{m-k-1}-1) + (2^{m-k-1}+1)$

$= 2.\ 2^{m-k-1} = 2^{m-k}$

Therefore $\gamma(Q_m) \geq 2^{m-k}$.

Subcase 2: (general case) $|S^0| = 2^{m-k-1} - r,\ r > 0$

We know $\gamma(Q_{m-1}) = 2^{m-k-1}$ and any minimum dominating set of Q_{m-1} contain 2^{m-k-1} elements and is also perfect.

So, $|S^0| = 2^{m-k-1} - r$ implies that, there exists at least r elements $u_1^0, u_2^0, \ldots, u_r^0$ in $V(Q_{m-1}^0)$ with $d(u_i^0, u_j^0) \geq 3$, such that elements of $N[u_1^0] \cup N[u_2^0] \ldots \cup N[u_r^0]$ are not dominated by any element of S^0 in Q_{m-1}^0. So to dominate these vertices in Q_m, rm vertices of $N[u_1^0] \cup N[u_2^0] \ldots \cup N[u_r^0]$ must be included in D and hence in S^1.

These rm elements dominate at most $r(m-1)(m-2)$ vertices of Q_{m-1}^1 other than the elements of $N[u_1^1] \cup N[u_2^1] \ldots \cup N[u_r^1]$.

So remaining vertices in Q_{m-1}^1 are $2^{m-1} - rm - r(m-1)(m-2)$

$= 2^{m-1} - rm^2 + 3rm - rm - 2r = 2^{m-1} - rm^2 + 2rm - 2r.$

To dominate these vertices at least $(2^{m-1} - rm^2 + 2rm - 2r)/m$ vertices are needed.

Hence $|S^1| \geq rm + (2^{m-1})/m - rm + 2r(m-1)/m$

$= 2^{m-k-1} + r(2-2/m) = 2^{m-k-1} + r(1+1-2/m) \geq 2^{m-k-1} + r$

Hence in this case also, $|D| = |S^0| + |S^1| \geq (2^{m-k-1} - r) + (2^{m-k-1} + r) = 2.\ 2^{m-k-1} = 2^{m-k}$

Therefore, $\gamma(Q_m) \geq 2^{m-k}$.

Hence in all cases $\gamma(Q_m) \geq 2^{m-k}$. -------------------------- (II)

From (I) and (II), $\gamma(Q_m) = 2^{m-k}$.

Note: Let D_1, D_2 be two disjoint perfect dominating sets of Q_{n-1}. Let D_1^0, D_2^1 be the corresponding dominating sets of $(Q_{n-1})^0$, $(Q_{n-1})^1$. Then $D_1^0 \cup D_2^1$ is an independent minimum dominating set for Q_n with cardinality 2^{n-k}.

Theorem 3.2 $\gamma_t(Q_m) = 2^{m-k} = \gamma_p(Q_m) = \gamma_i(Q_m)$, where $m = 2^k$.

Proof: $Q_m = Q_{m-1} \times K_2$. We can view Q_m as $V(Q_m) = V((Q_{m-1})^0) \cup V((Q_{m-1})^1)$.

$E(Q_m) = E((Q_{m-1})^0) \cup E((Q_{m-1})^1) \cup \{ u_i^0 u_i^1 : u_i^0 \in V((Q_{m-1})^0), u_i^1 \in V((Q_{m-1})^1)\}$, $i = 1, 2, \ldots, m-1$. Let D be a γ-dominating set of Q_{m-1}. Let D^0, D^1 be the corresponding dominating sets of Q_{m-1}^0, Q_{m-1}^1 respectively. $D^0 \cup D^1$ is a γ-dominating set of Q_m and is total. Therefore, $\gamma_t(Q_m) = 2 \times 2^{m-k-1} = 2^{m-k}$. We know that, $V(Q_{m-1})$ can be partitioned into 2^k sets each of which is the dominating set of Q_{m-1} containing 2^{m-k-1} elements and each of those dominating set is perfect in $Q_{m-1.}$ Let $(D_1)^0$ and $(D_2)^0$ be two disjoint

dominating sets in the domatic partition of $(Q_{m-1})^0$. Let $(D_2)^1 = \{u_i^1 : u_i^0 \in (D_2)^0\}$ then $(D_2)^1$ is a dominating set of $(Q_{m-1})^1$ and $(D_1)^0 \cup (D_2)^1$ is a minimum dominating set of Q_m and is independent. Therefore, $\gamma_i(Q_m) = 2\times 2^{m-k-1} = 2^{m-k}$. This proves the theorem.

Theorem 3.3 If $n = 2^k+s$ with $0 < s < 2^k-1$, then $2^{n-k-1} < \gamma(Q_n) \leq 2^{n-k}$.

Proof: $n = 2^k+s$, $0 < s < 2^k-1$. We have $\gamma(Q_n) \leq 2\gamma(Q_{n-1})$ for all n. As $m = 2^k$, we have $\gamma(Q_{m+s}) \leq 2\gamma(Q_{m+s-1}) \leq 2^2\gamma(Q_{m+s-2}) \leq\ldots \ldots\leq 2^s\gamma(Q_m) = 2^s \times 2^{m-k} = 2^{m+s-k} = 2^{n-k}$. Also, $\gamma(Q_n) \geq 2^n/(n+1)$.

Hence, $\gamma(Q_{m+s}) \geq 2^{m+s}/(m+s+1) > 2^{m+s}/(2m) = 2^{m+s}/(2^{k+1}) = 2^{m+s-k-1} = 2^{n-k-1}$. Therefore, $2^{n-k-1} < \gamma(Q_n) \leq 2^{n-k}$. But $s > 0$ and $s < 2^k-1$. Hence, $2^{n-k-1} < \gamma(Q_n) \leq 2^{n-k}$.

4. Independent Domination of Q_n

Theorem 4.1 $\gamma_i(Q_n) \leq 2^{n-k}$ for all n such that $2^k-1 \leq n \leq 2^{k+1}-2$.

Proof: When $n = 2^k-1 = m-1$, we know $\gamma(Q_{m-1}) = \gamma_i(Q_{m-1}) = \gamma_p(Q_{m-1}) = 2^{m-k-1} = 2^{n-k}$ and $V(Q_{m-1})$ can be partitioned into $m = 2^k$ dominating sets of cardinality 2^{m-k-1}. Therefore, $V(Q_{m-1}) = D_1 \cup D_2 \cup\ldots\cup D_m$, where each D_i is a perfect dominating set for Q_{m-1} and $D_i \cap D_j = \emptyset$ for $i \neq j$. Let $(D_1)^0$, $(D_2)^1$ be dominating sets of Q_{m-1}^0 and Q_{m-1}^1 respectively corresponding to the dominating sets D_1, D_2 of Q_{m-1}.

In general, if $n = 2^k+s$ for $0 \leq s \leq 2^k-2$, then Q_n has a vertex partition into 2^{s+1}copies of Q_{m-1} and the degree of each vertex in Q_n is $n = 2^k+s = m+s$. Therefore only s+1 vertex disjoint copies of Q_{m-1} are at distance one from a particular vertex x of Q_n. Hence we can take copies of an independent dominating set of Q_{m-1} in such a way that their union is an independent dominating set for Q_{m+s}. (since $s \leq 2^k-2 = m-2$ and there are 2^k independent mutually disjoint dominating sets for Q_{m-1}, this is always possible). Thus, $\gamma_i(Q_n) = \gamma_i(Q_{m+s}) \leq 2^{s+1}\gamma_i(Q_{m-1}) = 2^{s+1}2^{m-k-1} = 2^{n-k}$; that is, $\gamma_i(Q_n) \leq 2^{n-k}$ for all n such that $2^k-1 \leq n \leq 2^{k+1}-2$.

5.Connected and Total domination of Q_n

Theorem 5.1

If $2^k-1 \leq n \leq 2^{k+1}-2$, then $\gamma_c(Q_n) \leq 2^{n-k} + 2^{m-k} -2$, where $m = 2^k$.

Proof:

Case 1: $n = 2^k-1 = m - 1$.

Let D be a γ-dominating set of Q_{m-1}. We know $\gamma = \gamma_i = \gamma_p = |D| = 2^{m-k-1}$. The distance between any two elements in D is at least 3 and if $u \in D$, then there exists $v \in D$ such that $d(u, v) = 3$ in Q_{m-1}. Let $D = \{v_1, v_2, v_3, \ldots, v_t\}$, where $t = 2^{m-k-1}$.

Let D be the graph with vertex set D and whose edge set is defined by $< x, y > \in E(D)$ if and only if $d(x, y) = 3$ in Q_{m-1}. We claim that D is connected. If not V(D) can be partitioned into non-empty subsets D_1 and D_2 such that there is no edge in D between D_1 and D_2. This means that for any $x \in D_1$ and any $y \in D_2$, $d(x, y) \geq 4$ in Q_{m-1}. Let $d(D_1, D_2) = d(u_0, v_0) = d_0$, $u_0 \in D_1$, $v_0 \in D_2$. Let $u_0, x_1, x_2, \ldots, x_{d0} = v_0$ be a u_0, v_0 path of length d_0. If $x_2 \in N(u)$ for some $u \in D_1$ then $u, x_2, x_3, \ldots, x_{d0} = v_0$ is a path from D_1 to D_2 of length $d_0 - 1$, contradicting the minimality of d_0. If $x_2 \in N(v)$ for some $v \in D_2$, then u_0, x_1, x_2, v is a D_1, D_2 path of length $3 \leq d(D_1, D_2)$ in Q_{m-1}. Thus $x_2 \notin N(D_1) \cup N(D_2) = N(D)$, contradicting the fact that D is a dominating set . Thus D must be connected. Hence it has a spanning tree, with $|D| - 1$ edges. Now each edge $< x, y > \in D$ corresponds to an x, y path of length 3 in Q_{m-1}. So by adjoining to D two vertices per edge of D we obtain a connected dominating set S, of size $|D| + 2(|D| - 1) = (2^{m-k-1} + 2\times 2^{m-k-1}) - 2 = (3\times 2^{m-k-1}) - 2 = 3\times 2^{n-k} - 2$. This proves case 1.

Case 2: $\underline{n = 2^k} = m$.

Let D be a dominating set of Q_{m-1}. Let $D^0 = \{x^0 \in Q_m \mid x \in D\}$ and $D^1 = \{x^1 \in Q_m \mid x \in D\}$, and for each $x \in D$, x^0 and x^1 are adjacent in Q_m. We have seen in the proof of part (1) above that there is a connected dominating set S of $(Q_{m-1})^0$ which contains D^0 and whose size is $(3\times 2^{m-k-1}) - 2$. Since each vertex of D^1 is adjacent to vertex of $D^0 \subseteq S$, $S \cup D^1$ is connected. Since for i = 0, 1 D^i dominates $(Q_{m-1})^i$, $D^0 \cup D_1$dominates Qm. Hence so does $S \cup D^1$, which is thus a connected dominating set for Q_m. Its size is $|S| + |D^1| = (3\times 2^{m-k-1}) - 2 + 2^{m-k-1} = 2^2\times 2^{m-k-1} - 2 = 2^{m-k+1} - 2 = 2^{n-k+1} - 2$. Thus $\gamma_c(Q_n) \leq 2^{n-k+1} - 2$.

Case 3: $n = 2^k + i = m+i = (m-1) + (i+1)$

Consider Q_n, where $n = 2^k + i = m+i = (m-1)+(i+1)$, where $1 < i \leq 2^k - 2$. Q_n has 2^{i+1} vertex disjoint copies of Q_{m-1}. Name them $(Q_{m-1})^x$, for each i+1 dimensional Boolean vector x. Let D be a perfect dominating set of Q_{m-1}. Let D^x be the corresponding dominating sets of $(Q_{m-1})^x$. As in the proof of case 1, we can find a connected dominating set D of $(Q_{m-1})^0$ with cardinality $(3\times 2^{m-k-1} - 2)$. Therefore, $D \cup_x D^x$ is a connected dominating set of Q_n, where x ranges over all i+1 dimensional Boolean vectors.

Hence, $\gamma_c(Q_n) \leq (3\times 2^{m-k-1}) - 2 + \Sigma 2^{m-k-1} = (3\times 2^{m-k-1}) - 2 + (2^{i+1} - 1)\, 2^{m-k-1}$

$= (2^{i+1}+2)\, 2^{m-k-1} - 2 = [2^{n-k} + 2^{m-k}] - 2.$

Thus the result is true for $n = 2^k + i$, where $1 < i \leq 2^k - 2$.

This proves the theorem.

Theorem 5.2

(1) If $n = 2^k-1$, then $\gamma_t(Q_n) \leq 2^{n-k+1}$.

(2) If $2^k \leq n \leq 2^{k+1}-2$, then $\gamma_t(Q_n) \leq 2^{n-k}$.

Proof of (1): $n = 2^k-1$.

Let D be a γ-dominating set of Q_n. Then $|D| = 2^{n-k} = \gamma(Q_n)$. Let $D' = D + e_1$, where $e_1 = 100\ldots0$. Then $D \cup D'$ is a total dominating set and so $\gamma_t(Q_n) \leq |D \cup D'| = 2 \times |D| = 2^{n-k+1}$.

Proof of (2): $2^k \leq n \leq 2^{k+1}-2$.

Let $n = 2^k+i = m+i = (m-1)+(i+1)$, where $1 \leq i \leq 2^k-2$. Q_n has 2^{i+1} vertex disjoint copies of Q_{m-1}. Each can be denoted by $(Q_{m-1})^x$, where x is any i+1 dimensional Boolean vector. Let D be a perfect dominating set of Q_{m-1} and let D^x be the corresponding dominating sets of $(Q_{m-1})^x$. Therefore, $\cup_x D^x$ is a total dominating set of Q_n. Hence, $\gamma_t(Q_n) \leq 2^{i+1} \times 2^{m-k-1} = 2^{n-k}$. This completes the proof of the theorem.

Conjecture: We have $Q_n = Q_{n-1} \times K_2$. Any dominating set D of $Q_{n-1}{}^0$ dominates $|D|$ elements in $Q_{n-1}{}^1$ and the remaining vertices form a subgraph of $Q_{n-1}{}^1$ whose highest degree is n−2. Using the fact that the size of a minimum dominating set D of a graph G is bounded above by $\lceil |G| / (\Delta+1) \rceil$, we conjecture that when $2^k+1 \leq n \leq 2^{k+1}-2$,

$\gamma(Q_n) = \gamma(Q_{n-1}) + \lceil (2^{n-1} - \gamma(Q_{n-1}))/ (n-1) \rceil$.

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On The Chromatic Preserving Sets

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Abstract: Let G be a simple graph with vertex set V(G) and edge set E(G). A Set $S \subseteq V$ is said to be a chromatic preserving set or a cp-set if $\chi(\langle S\rangle) = \chi(G)$ and the minimum cardinality of a cp-set in G is called the chromatic preserving number or cp-number of G and is denoted by cpn(G). A cp-set of cardinality cpn(G) is called a cpn-set. A partition of V(G) is said to be a cp-partition, if each subset in the partition induces a chromatic preserving set (cp-set). The cp-partition number of a graph G is defined to be the maximum cardinality of a cp-partition of V(G) and is denoted by cppn(G). In this paper, cp-number and cp-partition number of some standard graphs are found. Some of the graphs for which cpn(G) = χ(G) are identified. Some Nordhaus-Gaddum type of results are obtained for cp-number and cp-partition number.

Keywords: Chromatic preserving set, chromatic preserving number, cp-partition, cp-partition number
MSC 2000: 05C15

1. Introduction

Graphs considered in this paper are finite, simple and undirected. For any graph G, the vertex set and edge set are denoted by V(G) and E(G) respectively.

A *clique* of a graph G is a maximal complete sub graph. The cardinality of a maximum clique is called the ***clique number*** and is denoted by $\omega(G)$. A ***wheel*** W_n is obtained by joining each vertex of C_{n-1} to an isolated vertex. If S is a non empty subset of the vertex set of a graph G, then the sub graph of G induced by S is the graph with vertex set S and edge set consisting of all those edges of G with both the end vertices in S and is denoted by <S>. A ***block*** of a graph G is a maximal, 2-connected sub graph of G. A graph G is a ***block graph*** if and only every block of G is a complete graph. A graph G is said to be a ***perfect graph*** if $\chi(H) = \omega(H)$ for all induced sub graphs H of G . A graph G is ***chordal*** or ***triangulated*** if every cycle of length greater than three has a chord. A set of vertices in a graph G is ***independent*** if no two of them are adjacent in G. The maximum cardinality among such independent sets is called the ***independence number*** of G and is and denoted by $\beta_o(G)$. An independent set of edges of G has no two of its edges adjacent and the maximum cardinality of such a set is the ***edge independence number*** $\beta_1(G)$.

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A matching in a graph is a set of independent edges and a perfect matching is a set of independent edges such that each vertex is an end vertex of some edge.

A set $S \subseteq V$ is a ***dominating set*** of G if for each $u \in V - S$, there exists a vertex $v \in S$ such that u is adjacent to v. The minimum cardinality of a dominating set in G is called the ***domination number*** and is denoted by $\gamma(G)$. A dominating set $S \subseteq V$ of G is a ***total dominating set*** if <S> has no isolated vertices. The minimum cardinality of a total dominating set in G is called the ***total domination number*** and is denoted by $\gamma_t(G)$. A dominating set $S \subseteq V$ of G is a ***connected dominating set*** if <S> is a connected subgraph of G. The minimum cardinality of a connected dominating set in G is called the ***connected domination number*** and is denoted by $\gamma_c(G)$.

A ***k-coloring*** of a graph G is a labeling $f : V \longrightarrow \{1,2,\ldots,k\}$. The labels are colors; the vertices with color i form a color class. A k-coloring is ***proper*** if $xy \in E$ implies $f(x) \neq f(y)$. A graph G is ***k-colorable*** if it has a proper k-coloring. The ***chromatic number*** $\chi(G)$ is the minimum k such that G is k-colorable. If $\chi(G) = k$, then G is said to be ***k-chromatic***. If $\chi(G) = k$, but $\chi(G) < k$ for every proper sub graph H of G, then G is said to be a ***k-color-critical graph***. A graph G is said to be a ***vertex-color-critical graph*** or ***k-critical graph*** if $\chi(G - u) < \chi(G)$ for every $u \in V$. Critical graphs were defined by Dirac [1]. In the literature, there are many questions posed by mathematicians on critical graphs. The book by Jensen and Toft [4] lists all famous problems on critical graphs. The k-critical graphs for k = 1, 2 and 3 are K_1, K_2 and odd cycles, respectively. For $k \geq 4$, the k-critical graphs have not been characterized. Ordinarily, it is extremely difficult to determine whether a given graph is critical; however every k-chromatic graph $k \geq 2$ contains a k-critical sub graph. In fact, if H is any smallest (in terms of number of vertices) induced sub graph of G such that $\chi(G) = \chi(H)$, then H is critical. Also not much is known on how to find the smallest critical sub graph of a non critical graph. Hermann [3] made an attempt to propose new exact algorithms for finding the chromatic number of a graph G. The algorithm attempts to determine the smallest possible induced sub graph H of G, which has same chromatic number as G. As mathematicians are more interested in rigorous proof techniques, in this paper we made an attempt to find the smallest subset of a vertex set, which induces a critical graph having the same chromatic number of the given graph. We define a vertex subset satisfying the above condition as a ***chromatic preserving set*** (or cp-set) and the minimum chromatic preserving set as ***cpn-set***. In a real life situation, finding a cpn-set gives a feasible solution to number of problems. For an example, in an Organization/Institution, the management is interested in forming a team from employees/students to train them in different specified skills so that (i) the members are

reachable from each other (by reachable we mean that no sub team is isolated from the main team) (ii) there is at least one member for each specified skill (iii) one member is selected only for one skill (iv) two members with direct association are not selected for the same training (v) the minimum number of different skills that can be given to the group and (vi) the size of the team is as small as possible. Now, if a graph is drawn with the vertices representing the employees or students and an edge is drawn if an association exists between any two members. If different skills are marked by different colors, then the proper coloring of the graph satisfies (iii) and (iv). Finding the chromatic number of the graph satisfies condition (v). Finally, a minimal cp-set satisfies conditions (i) and (ii), and a cpn-set satisfies condition (vi).

Unless otherwise mentioned any graph considered in this paper is a (p, q)-graph. Definitions not given may be referred to [2] and [5].

2. Prior Results

Theorem 2.1 ([5], pp 177). If G is a k-color-critical graph, then $\delta(G) \geq k - 1$.

Theorem 2.2 ([2], pp 129). For any graph G, $\chi(G) + \chi(\overline{G}) \leq p + 1$.

Proposition 2.3 Graphs G and $\overline{G}$ are bipartite if and only if $G = C_4, P_3, P_4$, or $2K_2$.

Proof: If G has a vertex u such that $d(u) > 2$, then in $\overline{G}$, the neighbors of u form a 3-cycle. If $diam(G) > 3$, then in $\overline{G}$, a 3-cycle is induced. In both cases contradiction arises.

3. Main Results

3.1. Chromatic preserving sets in graphs

Definition 3.1.1. A set $S \subseteq V$ is said to be a chromatic preserving set or a cp-set if $\chi(\langle S \rangle) = \chi(G)$ and the minimum cardinality of a cp-set in G is called the chromatic preserving number or cp-number of G and is denoted by cpn(G). A cp-set of cardinality cpn(G) called cpn-set.

Example 3.1.2. The graph G given in Figure 1(Appendix II) is 3-chromatic graph with 10 vertices and 15 edges and it does not have a 3-cycle. From that Figure, it is clear that the minimum odd cycle is of length 5 and hence, cpn(G) = 5.

Observation 3.1.3. Properties of a minimal cp-set:

i. If $\chi(G) \geq 3$, then cpn-set induces a 2-connected vertex-color-critical graph.
ii. If G is connected, then cpn(G) = p if and only if G is vertex-color-critical.
iii. There does not exist any disconnected graph G such that cpn(G) = p.
iv. cpn(G) = 1 if and only if $G = nK_1$, $n \geq 1$.
v. G is a bipartite graph if and only if cpn(G) = 2.
vi. If G is 3-chromatic, then cpn(G) = $g_o(G)$.
vii. If cpn(G) = 3, then G is 3-chromatic.
viii. For any non-trivial graph G, which is neither vertex-color-critical nor totally disconnected, $2 \leq cpn(G) \leq p - 1$.
ix. For a disconnected graph G with k < p components, $cpn(G) \leq p - k + 1$.
x. If $\chi(G - u) < \chi(G)$ for a vertex $u \in V$, then u is in every minimal cp-set of G and conversely. Similarly, if $\chi(G - e) < \chi(G)$ for an edge $e \in E$, then e is in every minimal cp-set of G and conversely.
xi. If $\chi(G - u) = \chi(G)$, then $cpn(G - u) \geq cpn(G)$.
xii. If $cpn(G) \leq 4$, then G is a perfect graph.

The following Proposition gives the cp-number of some standard graphs.

Proposition 3.1.4.

i. $cpn(K_n) = n$;

ii. $cpn(C_n) = \begin{cases} 2, & \text{if n is even} \\ n, & \text{if n is odd;} \end{cases}$

iii. $cpn(W_n) = \begin{cases} 3, & \text{if n is even} \\ n, & \text{if n is odd;} \end{cases}$

Proposition 3.1.5. For any graph G,

i. $\omega(G) \leq \chi(G) \leq cpn(G)$.

ii. $\chi(G) = cpn(G)$ if and only if cpn-set induces a complete graph.

Proof:

i) Follows trivially.

ii) Suppose a cpn-set induces a complete graph and S is a cpn-set of G. Then $<S> = K_r$ for some r and hence, cpn(G) = r. Further $\chi(G) = \chi(<S>) = r$ and the result follows. Conversely, suppose $cpn(G) = \chi(G)$ and S is a cpn-set of G. Therefore, $|S| = cpn(G) = \chi(<G>) = \chi(<S>)$. Hence, S induces a complete graph.

Proposition 3.1.6. If G is a perfect graph, then $cpn(G) = \chi(G) = \omega(G)$.

Proof: If G is perfect, then $\omega(G) = \chi(G)$. Then a cpn-set induces a complete graph and the result follows from Proposition 3.1.5(ii).

Proposition 3.1.7. If G is a disconnected graph with components $G_1, G_2, \ldots, G_k$, then cpn(G) $= \min_i \{cpn(G_i) \mid \chi(G_i) = \chi(G)\}$.

Proposition 3.1.8. If S is a cpn-set of a graph G and $\chi(G) = k \geq 3$, then $\delta(\langle S\rangle) \geq k - 1$.

Proof: Since <S> is vertex-color-critical, from Theorem 2.1, the result follows.

Proposition 3.1.9. If a connected graph G has a dominating cpn-set and $\gamma(G) \geq 2$, then $\gamma(G) \leq \gamma_t(G) \leq \gamma_c(G) \leq cpn(G)$.

Proposition 3.1.10. A graph G is 3-chromatic with $g_o(G) = 5$ or a 5-chromatic graph containing K_5 as a maximal complete graph G if and only if cpn(G) = 5.

Proof: Necessary part is trivial. Now suppose cpn(G) = 5 and G is a k-chromatic graph. Clearly, $3 \leq k \leq 5$.

<u>Claim</u> : $k \neq 4$.

Suppose k = 4. Let S = {a, b, c, d, e} be a cpn-set of G. Since <S> is 4 -chromatic, exactly two of the vertices are of same color and the remaining three vertices are of different colors. Then one of the vertices of same color must be adjacent to all the remaining 3 colors. This adjacency induces a K_4, which consequently implies cpn(G) = 4, a contradiction.

<u>Case i</u>: k = 3.

From Observation 3.1.3(vi), cpn(G) = $g_o(G)$ and the result follows.

<u>Case ii:</u> k = 5.

Then cpn(G) = $\chi(G)$ and by Proposition 3.1.5(ii), cpn-set induces a complete graph and hence, G contains K_5 as a maximal complete sub graph.

Proposition 3.1.11. For a graph G,

i. cpn(G) = cpn $(\overline{G})$ = 1 if and only if G = K_1.

ii. cpn(G) = cpn $(\overline{G})$ = 2 if and only if G = P_3, $K_2 \cup K_1$, P_4, C_4, $2K_2$.

Proposition 3.1.12. If cpn(G) = cpn$(\overline{G})$ = 3, then $\chi(G) = 3$ and $p \geq 5$.

Proof: If cpn(G) = 3, then $\chi(G) = 3$ and G has a 3-cycle say u-v-w-u. By similar argument, $\overline{G}$ has a 3-cycle say x-y-z-x. Then x, y, z are independent vertices in G. Hence, at least 2 vertices of x, y, z cannot be in the 3-cycle u-v-w-u. Thus $p \geq 5$.

Theorem 3.1.13. For a connected graph G, cpn(G) = cpn($\overline{G}$) = 3 if and only if G and $\overline{G}$ are 3-chromatic and contain at least one of the sub graphs in the family of graphs given in Figure 2 (Appendix II) as induced sub graph.

Proof: Necessary part is trivial. Now suppose cpn(G) = cpn($\overline{G}$) = 3. Then G and $\overline{G}$ are 3-chromatic and hence, both G and $\overline{G}$ contain 3-cycles. Let a-b-c-a and d-e-f-d be 3-cycles in G and $\overline{G}$ respectively. Clearly, {d, e, f} forms an independent set in G and {a, b, c} forms an independent set in $\overline{G}$. Two cases arise. Here, the case is being discussed for the graph G. Similar discussion can be had for $\overline{G}$ also. Let ß be the family of graphs given in Figure 2 (Appendix II).

<u>**Case i:**</u> $\{a, b, c\} \cap \{d, e, f\} \neq \phi$.

Clearly, $|\{a, b, c\} \cap \{d, e, f\}| = 1$. Without loss of generality, let a = f. Then {a, d, e} forms an independent set in G. Let S = {a, b, c, d, e}. Then $deg_{<S>}(a) = 2$, $2 \leq deg_{<S>}(b) \leq 4$ and $2 \leq deg_{<S>}(c) \leq 4$. Table 1 (a) (Appendix I) lists the graphs <S> for the various values of deg(b) and deg(c) with deg(a) = 2 in <S>.

<u>**Case ii:**</u> $\{a, b, c\} \cap \{d, e, f\} = \phi$.

Let H = <{a, b, c, d, e, f}>. In H, the following facts are observed.

<u>**Fact 1:**</u> At most two vertices of {a, b, c} can be adjacent to the same vertex of {d, e, f}. Otherwise K_4 is induced.

<u>**Fact 2:**</u> Each vertex of {a, b, c} is adjacent to a vertex of {d, e, f}.

Otherwise $\beta_o(H) = 4$. Then K_4 is induced in $\overline{G}$, a contradiction to $\overline{G}$ is a 3-chromatic graph.

<u>**Fact 3:**</u> A vertex of {d, e, f} can be adjacent to at most two vertices of {a, b, c}. Otherwise K_4 is induced.

<u>**Sub case i:**</u> H is disconnected.

<u>**Claim:**</u> H has only one isolated vertex.

Suppose $H = K_3 \cup 3K_1$. Then similar argument as in Fact 2 leads to a contradiction. So suppose H has 2 isolated vertices say d and e. From Fact 2 each vertex of {a, b, c} is adjacent to vertex f and inducing K_4, a contradiction. Hence, the claim holds.

Let d be the isolated vertex of H. then from Fact 2, $3 \leq deg(a), deg(b), deg(c) \leq 4$ and from Fact 3, $1 \leq deg(e), deg(f) \leq 2$. Then the following fact is observed.

<u>**Fact 4:**</u> At most one vertex of {a, b, c} can be of degree 4.

Suppose deg(a) = deg(b) = 4. Then as deg(c) ≥ 3, a, b, c are adjacent to the same vertex of {d, e, f}, contradicting Fact 1.

If deg(a) = deg(b) = deg(c) = 3, then graph(h) is induced. Suppose deg(a) = deg(b) = 3, deg(c) = 4. From Fact 1, graph (i) is induced.

Sub case ii: H is connected.
From Fact 3, 1 ≤deg(d), deg(e), deg(f) ≤ 2 and from Fact 2, 3 ≤deg(a), deg(b), deg(c)≤ 5. Then the following facts are observed.
Fact 5: At most one vertex of {a, b, c}can be of degree 5.
Suppose deg(a) = deg(b) = 5. Then from Fact 1, deg(c) = 2 contradicting deg(c) ≥ 3.
Fact 6: Two vertices of {a, b, c} are of degree 4 and one vertex of {a, b, c} is of degree 5 cannot hold. Suppose fact 6 does not hold. Then sum of the degrees of vertices d, e and f is 7, a contradiction to the fact that sum of the degrees is at most 6 as 1 ≤ deg(d), deg(e), deg(f) ≤ 2.
Table 1 (b) (Appendix I) lists the graph H for the various values of deg(a), deg(b) and deg(c).

Proposition 3.1.14. If for a graph G, cpn(G) = cpn($\overline{G}$) = 3, then $\beta_o(G) = \beta_o(\overline{G}) = 3$.
Proof: cpn($\overline{G}$) = 3 implies that $\beta_o(G) \geq 3$. If $\beta_o(G) \geq 4$, then $\overline{G}$ contains K_4 as an induced sub graph, a contradiction. Hence, $\beta_o(G) = 3$. Similarly, $\beta_o(\overline{G}) = 3$.

Definition 3.1.15. A graph G is called a well colored graph if all minimal cp-sets have the same cardinality.

Observation 3.1.16.
i. Totally disconnected graph is a well colored graph.
ii. Bipartite graph is a well colored graph.
iii. Triangulated graph is a well colored graph.
iv. Block graph is a well colored graph.

Proposition 3.1.17. If G is a block graph with a single cut vertex, then $\overline{G}$ is a well colored graph.
Proof: Let u be the cut vertex of G and $G - u = G_1 \cup G_2 \cup \ldots \cup G_k$. Let $|V(G_i)| = n_i$, $1 \leq i \leq k$. Then $\overline{G} = K_{n_1, n_2, \ldots, n_k}$ and the result follows.

3.2. Cp-partition of graphs

Two new parameters cp-partition and cp-partition number are defined in this section.

Definition 3.2.1. A partition of V(G) is said to be a cp-partition, if each subset in the partition induces a chromatic preserving set (cp-set). The cp-partition number, cppn(G) is defined to be the maximum cardinality of a cp-partition of V(G).

Proposition 3.2.2.

i. $cppn(K_n) = 1$;

ii. $cppn(nK_1) = n$;

iii. $cppn(K_{1,n}) = 1$;

iv. $cppn(K_{m,n}) = \min\{m, n\}$, $m, n \geq 2$;

v. $cppn(C_n) = \begin{cases} n/2, & \text{if n is even} \\ 1, & \text{if n is odd;} \end{cases}$

vi. $cppn(W_n) = 1$.

Observation 3.2.3.

i. If G is a vertex-color-critical graph, then cppn(G) = 1. The converse need not be true.

ii. For each pair of positive integers n and r, there exists a graph G with cpn(G) = n and cppn(G) = r.

Proof:

i) Consider the graph $G = K_n - e$ for any n. Then cpn(G) = n - 1, and hence, cppn(G) = 1. Clearly, G is not vertex-color-critical.

ii) A complete n-partite graph $K_{r,r,\ldots,r}$ satisfies the required properties.

Proposition 3.2.4.

i. If G is a bipartite graph, then $cppn(G) = \beta_1(G)$.

ii. If G is a connected graph with $\chi(G) = k$, then $cppn(G) \leq \frac{p}{k}$.

iii. If G is not totally disconnected, then $cppn(G) \leq \frac{p}{2}$.

iv. cppn(G) = p if and only if $G = pK_1$.

v. If G is bipartite, then $cppn(G) \leq \frac{p}{2}$ and the equality holds if and only if G has a perfect matching.

Proof: Trivial.

Proposition 3.2.5. For a connected graph G,

i. cpn(G) = cppn(G) = 1 if and only if $G = K_1$.

ii. cpn(G) = cppn(G) = 2 if and if and only if $G = P_5$, P_4, C_4 or $G \in ß$, where ß is the family of graphs given in Figure 3 (Appendix II).

3.3. Some Nordhaus-Gaddum type of results

Proposition 3.3.1. For a perfect graph G, $cpn(G) + cpn(\overline{G}) \leq p + 1$.

Proof: G is perfect if and only if $\overline{G}$ is perfect. From Proposition 3.1.6, cpn(G) = χ(G) and cpn($\overline{G}$) = $\chi(\overline{G})$. Hence, cpn(G) + cpn($\overline{G}$) = χ(G) + $\chi(\overline{G}) \leq p + 1$.

Proposition 3.3.2. If G is vertex-color-critical, then cpn(G) + cpn($\overline{G}$) = p + 1 if and only if G is complete.

Proof: If G is complete, G = K_p and hence, $\overline{G}$ = pK_1. Hence, χ(G)= cpn(G)=p; $\chi(\overline{G})$=cpn($\overline{G}$)=1 and the result follows. Suppose cpn(G) + cpn($\overline{G}$) = p +1. Since G is vertex-color-critical, cpn(G) = p. Hence, cpn($\overline{G}$) = 1. Thus, $\overline{G}$ = pK_1.Therefore, G = K_p.

Proposition 3.3.3. If G is a perfect, then cpn(G) + cpn($\overline{G}$) $\leq$ p + 1.

Proof: If G is perfect, then $\overline{G}$ is perfect. From Proposition 3.1.6., cpn(G) = χ(G) and cpn($\overline{G}$) = $\chi(\overline{G})$. Then the result follows from Theorem 2.2.

Proposition 3.3.4. If G is bipartite and $\overline{G}$ is disconnected, then cpn(G) + cpn($\overline{G}$) = p + 1 if and only if G = $K_{1,p-1}$, p $\geq$ 3.

Proof: Necessary condition is trivial. Suppose cpn(G) + cpn($\overline{G}$) = p + 1. Since cpn(G) = 2, cpn($\overline{G}$) = p - 1. As $\overline{G}$ is disconnected, one component $\overline{H}$ of $\overline{G}$ has at least p - 1 vertices and $\chi(\overline{H}) = \chi(\overline{G})$. Since $\overline{G}$ has p vertices, and at least two components, $\overline{H}$ has exactly p - 1 vertices. Thus, $\overline{G}$ = $\overline{H} \cup K_1$. Consequently G = H + K_1. Since G is bipartite and K_1 is joined to all vertices of H, H must be a null graph. Hence, G = $K_{1,\ p-1}$.

Proposition 3.3.5. If G and $\overline{G}$ are 3-chromatic, then cpn(G) + cpn($\overline{G}$) = 2p if and only if G = C_5.

Proof: Necessary part is trivial. Suppose cpn(G) + cpn($\overline{G}$) = 2p. Then cpn(G) = cpn($\overline{G}$) = p. Therefore, G and $\overline{G}$ are vertex-critical graphs and hence, they are odd cycles. Clearly G, $\overline{G}$ $\neq$ C_3. Suppose G $\neq$ C_5. Then G = C_{2r+1}, r $\geq$ 3. This implies β_o(G) $\geq$ 3 and hence, $g_o(\overline{G})$ = 3, a contradiction.

Proposition 3.3.6. If G is a vertex-color-critical graph, then cpn(G) + cpn($\overline{G}$) = p + 1 if and only if G is complete.

Proof: If G is complete, then $\overline{G}$ is totally disconnected graph and thus, the result follows. Suppose

$cpn(G) + cpn(\overline{G}) = p + 1$. Since G is vertex-color-critical, $cpn(G) = p$ and therefore, $cpn(\overline{G}) = 1$. Thus, $\overline{G}$ is a 1-chromatic graph. Hence, $\overline{G}$ is either a trivial graph or a totally disconnected graph and hence, G is complete.

Observation 3.3.7. There exist k-chromatic graphs G and $\overline{G}$ such that $cpn(G) + cpn(\overline{G}) = 2k$.

Example 3.3.8. Let G be a graph obtained from K_k by adding pendant edges added at each vertex of K_k. Then both the graphs G and $\overline{G}$ are p-chromatic and contain K_k as an induced sub graph. Therefore, $cpn(G) = cpn(\overline{G}) = p$.

Proposition 3.3.9. If G and $\overline{G}$ are bipartite, then $cppn(G) + cppn(\overline{G}) = p$ if and only if $G = C_4$, P_4, or $2K_2$.

Proof: Necessary condition is trivial. Suppose $cppn(G) + cppn(\overline{G}) = p$. From Proposition 2.3, G and $\overline{G}$ are bipartite if and only if $G = C_4$, P_3, P_4, or $2K_2$. If $G = C_4$, P_3, $2K_2$ or P_4, then $\overline{G} = 2K_2$, $K_2 \cup K_1$, C_4, and P_4 respectively. If $G = P_3$, then $cppn(G) = cppn(\overline{G}) = 1$, a contradiction. If $G = C_4$, P_4, or $2K_2$, then $cppn(G) = cppn(\overline{G}) = 2$, and the result follows.

Proposition 3.3.10. If G is a vertex-color-critical graph, then $cppn(G) + cppn(\overline{G}) = p + 1$ if and only if G is complete.

Proof: Let G be a complete graph. Then $cpn(G) = p$ and $cpn(\overline{G}) = 1$.Therefore, $cppn(G) = 1$, $cppn(\overline{G}) = p$ and hence, $cppn(G) + cppn(\overline{G}) = p + 1$. Conversely suppose $cppn(G) + cppn(\overline{G}) = p+1$. Since G is vertex-color-critical graph, $cpn(G) = p$ and hence, $cppn(G) = 1$ This implies $cppn(\overline{G}) = p$. Therefore, $\overline{G}$ is a totally disconnected graph and hence, G is complete.

Proposition 3.3.11. If G is neither complete nor totally disconnected, then $cppn(G) + cppn(\overline{G}) \leq p$.

Proof: Since G is not totally disconnected, $cppn(G) \leq \frac{p}{2}$. As G is not complete, $\overline{G}$ is not totally disconnected and hence, $cppn(\overline{G}) \leq \frac{p}{2}$. Hence, $cppn(G) + cppn(\overline{G}) \leq p$.

4. Problems

1. Is it possible to characterize all k-chromatic graphs G and $\overline{G}$ of order p such that cpn(G) + cpn($\overline{G}$) = 2p.
2. For a k-chromatic graph G satisfying the condition in problem 1, what is the maximum number of edges that can be added to G so that the property is maintained.

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Appendix I

Table 1(a)

$\deg_{<S>}(b)$	$\deg_{<S>}(c)$	<S>
2	2	(a)
2	3	(b)
2	4	(c)
3	3	(d), (e)
3	4	(f)
4	4	(g)

Table 1(b)

$\deg_H(a)$	$\deg_H(b)$	$\deg_H(c)$	H
3	3	3	(j)
3	3	4	(k), (l)
3	3	5	(m)
3	4	4	(n), (o)
3	4	5	(p)
4	4	4	(q)

Appendix II

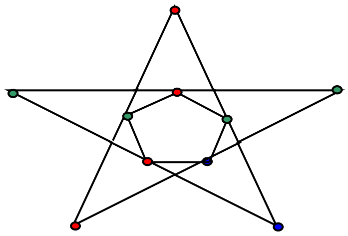
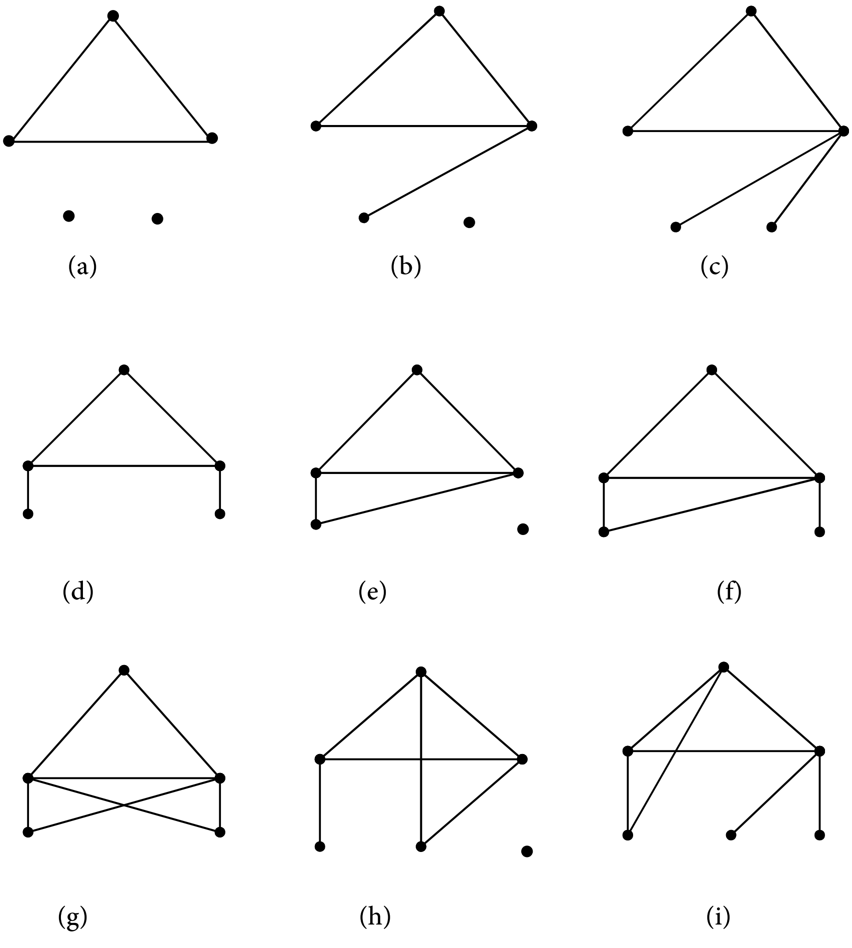


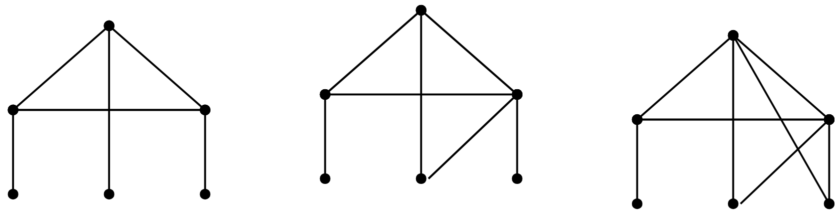
Figure 1



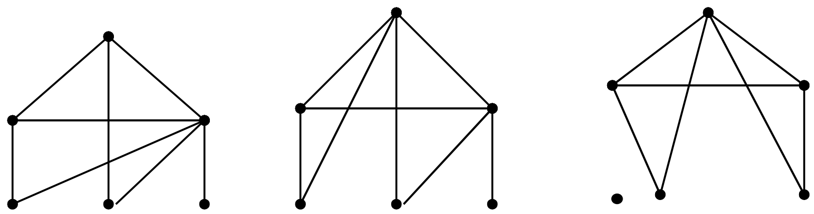
(a) (b) (c)

(d) (e) (f)

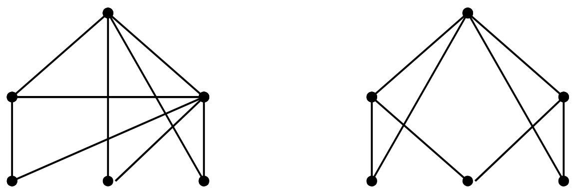
(g) (h) (i)



(j) (k) (l)

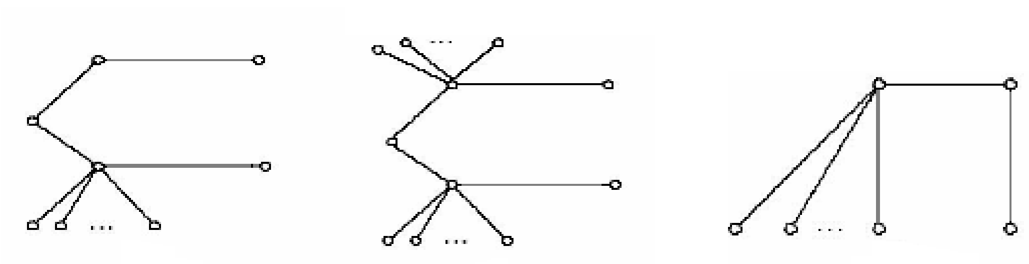


(m) (n) (o)

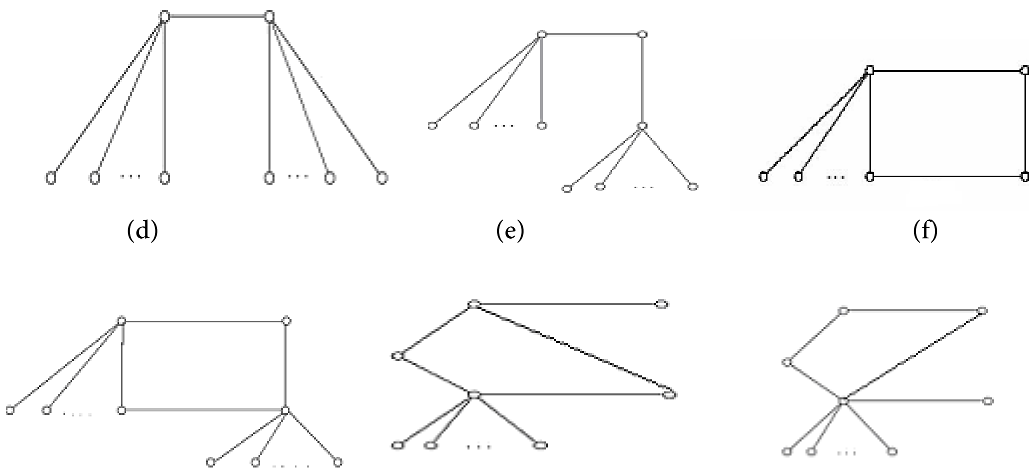


(p) (q)

Figure 2



(a) (b) (c)



(d) (e) (f)

(g) (h) (i)

Figure 3

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Acyclic Weak Convex Domination in Graphs

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Abstract: *In a graph G = (V, E), a set $D \subset V$ is a weak convex dominating(WCD) set if each vertex of V-D is adjacent to at least one vertex in D and $d_{<D>}(u,v) = d_G(u,v)$ for any two vertices u, v in D. A weak convex dominating set D, whose induced graph <D> has no cycle is called acyclic weak convex dominating (AWCD) set. The domination number γ_{ac} (G) is the smallest order of a acyclic weak convex dominating set of G and the codomination number of G, written $\gamma_{ac}(\overline{G})$, is the acyclic weak convex domination number of its complement. In this paper we found various bounds for these parameters and characterized the graphs which attain these bounds.*

Keywords: *domination number, distance, eccentricity, radius, diameter, self-centered, neighbourhood, weak convex dominating set, acyclic weak convex dominating set.*

1. Introduction

Finding a delay preserving sub network that can communicate with all the nodes of a communication network is a pioneer problem in communication network models. This delay preserving substructure performs with good tolerant index. This structure is studied in detail with the use of *Weak Convex Dominating* (W.C.D) set by considering the underlying graph of the communication network in [4]. Also the distance preserving dominating set concept will be useful in identifying the delay preserving sub network which can cover entire network as connected centre location of the given network. These concepts can be applied also to management and social network for similar applications. In this paper we concentrate on these sub structure with minimum number of links by defining a set called *Acyclic Weak Convex Dominating* (A.W.C.D) set.

Graphs discussed in this paper are undirected and simple. Unless otherwise stated the graphs which we consider are connected graphs only. For a graph G, let V(G) and E(G) denote its vertex and edge set respectively. The *degree* of a vertex v in a graph G is denoted by $\deg_G(v)$. The length of any shortest path between any two vertices u and v of a connected graph G is called the *distance between u and v* and is denoted by $d_G(u, v)$. The distance between two vertices in different components of a disconnected graph is defined to be ∞. For a connected graph G, the *eccentricity* $e_G(v) = \max\{d_G(u, v): u \in V(G)\}$. If there

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is no confusion, we simply use the notion deg(v), d(u, v) and e(v) to denote degree, distance and eccentricity respectively for the concerned graph. The minimum and maximum eccentricities are the *radius* and *diameter* of G, denoted r(G) and diam(G) respectively. When these two are equal, the graph is called *self-centered* graph with radius r, equivalently is *r self-centered*. A vertex u is said to be an *eccentric vertex* of v in a graph G, if d(u, v) = e(v). In general, u is called an eccentric vertex, if it is an eccentric vertex of some vertex. For $v \in V(G)$, the *neighbourhood* $N_G(v)$ of v is the set of all vertices adjacent to v in G. The set $N_G(v) = N_G(v) \cup \{v\}$ is called the *closed neighbourhood* of v. A set S of edges in a graph is said to be *independent* if no two of the edges in S are adjacent. An edge e = (u, v) is a *dominating edge* in a graph G if every vertex of G is adjacent to at least one of u and v.

The concept of domination in graphs was introduced by Ore[13]. A set $D \subseteq V(G)$ is called *dominating set* of G if every vertex in V(G)-D is adjacent to some vertex in D. D is said to be a *minimal* dominating set if D-{v} is not a dominating set for any $v \in D$. The *domination number* γ(G) of G is the minimum cardinality of a dominating set. We call a set of vertices a γ-set if it is a dominating set with cardinality γ(G). Different types of dominating sets have been studied by imposing conditions on the dominating sets. A dominating set D is called *connected* (*independent*) dominating set if the induced subgraph <D> is connected (independent). D is called a *total dominating set* if every vertex in V(G) is adjacent to some vertex in D.

A cycle D of a graph G is called a *dominating cycle* of G, if every vertex in V – D is adjacent to some vertex in D. A dominating set D of a graph G is called a *clique dominating set* of G if <D> is complete. A set D is called an *efficient dominating set* of G if every vertex in V – D is adjacent to exactly one vertex in D. A set $D \subseteq V$ is called a *global dominating set* if D is a dominating set in G and $\overline{G}$. A set D is called a *restrained dominating set* if every vertex in V(G)-D is adjacent to a vertex in D and another vertex in V(G)-D. A set D is a *weak convex dominating set* if each vertex of V-D is adjacent to at least one vertex in D and the distance between any two vertices u and v in the induced graph <D> is equal to that of those vertices u and v in G. By $\gamma_c, \gamma_i, \gamma_t, \gamma_o, \gamma_k, \gamma_e, \gamma_g, \gamma_r$ and γ_{wc}, we mean the minimum cardinality of a connected dominating set, independent dominating set, total dominating set, cycle dominating set, clique dominating set, efficient dominating set, global dominating set and restrained dominating set respectively.

In this paper we introduce a new dominating set called acyclic weak convex dominating set of a graph through which we studied the properties of the graph such as variation in radius and diameter of the graph. We find upper and lower bounds for the new domination number in terms of various already known parameters. Also we studied several interesting properties like Nordhaus-Gaddum type results relating the graph and its complement.

2. Prior Results

Theorem 2.1:[14]

Let G be any graph and D be any dominating set of G. Then | V-D | $\leq \sum_{u \in V(D)} \deg(u)$ and equality holds in this relation if and only if D has the following properties :

(i) D is independent.

(ii) For every $u \in$ V-D, There exist a unique vertex $v \in$ D such that N(u) $\cap$ D = {v}.

Theorem 2.2:[3]

For any tree T of order p $\geq$ 3, γ_c(S(T))= 2p-e-1, where 'e' denotes the number of pendent vertices of T.

3. Main Results

3.1 Existence of Acyclic Weak Convex Dominating sets:

Definition3.1:

A Weak convex dominating (W.C.D) set is said to be an *Acyclic weak convex dominating* (A.W.C.D) set, if D is acyclic. The cardinality of the minimum A.W.C.D set is denoted by γ_{ac}.

Proposition 3.1:

Let G be a unicyclic graph having a cycle C_n of length less than or equal to 6. Then G does not have any A.W.C.D set, if one of the following hold:

(i) there exists a cycle C_n, where n < 6 such that at least $\lfloor n/2 \rfloor$ + 2 number of vertices of C_n, each of which dominates uniquely some vertex in G

(ii) there exists a C_6 such that n/3 number of vertices of C_n each of which dominate uniquely some vertex in G.

Theorem 3.1:

Let G be any graph having girth $\geq$ 7, then it doesn't have A.W.C.D set.

Proof:

Let G be a graph having girth $\geq$ 7. Let D be a γ_{wc} set of G. We know that γ_{wc} (G) $\leq$ p. If γ_{wc}(G) = p, then G doesn't have A.W.C.D set. If γ_{wc}(G) $\neq$ p, then γ_{wc} (G) < p. This

implies that $| V\text{-}D | \geq 1$. Let $u \in V\text{-}D$. If two vertices of D dominate u, then as D is convex there must exists a C_3 or C_4 in G. Therefore, only one vertex of D dominates u. If $\deg(u) \neq 1$,then there exists another vertex $v \in V\text{-}D$ such that u and v are adjacent. And also u and v are not dominated by the same vertex of D (if possible, then C_3 arises). Therefore, u and v are dominated by two different vertices say some u' and v' of D respectively. This implies that $d(u', v') \leq 3$ (since length of the path $u'uvv' = 3$). This implies that there must be a C_4 or C_5 or C_6 exist in G, which is a contradiction to $g(G) \geq 7$. Hence V-D contains only the pendent vertices. That is all the vertices other than pendent vertices are in D. Hence, G does not have any A.W.C.D set.

Proposition 3.2:

Let G be a graph having a geodesic cycle C_n, where $n \geq 7$ does not have any A.W.C.D set.

Proof:

Let G be a graph on n vertices having C_p, where $p \geq 7$ as a geodesic cycle. For $n = 7$, $G = C_7$ which doesn't have any A.W.C.D set. Assume that for $n \geq 8$, the graph G on n vertices having C_p, where $p \geq 7$, has no A.W.C.D set.

To prove for a graph on $(n + 1)$ vertices, let G be a graph on $n+1$ vertices such that it has an acyclic dominating set D in G. Let $u \in V - D$ and $G' = G - u$. Without loss of generality, let us assume that u lies on a cycle of length at most 6. Clearly, by induction hypothesis, G' has no acyclic dominating set, which is a contradiction to the assumption that D is an acyclic dominating set for both G and G'. Thus the proposition is true.

Proposition 3.3:

Let G be a self-centred graph of diameter 2. Then γ_{ac} exists if and only if there exists a dominating set isomorphic to $K_{1,n}$.

Proof :

Let G be a self-centred graph of diameter 2. Suppose γ_{ac} exists in G. Let D be an A.W.C.D set in G. Then the induced sub graph <D> induced by D must be a tree. Clearly, $|D|$ is not equal to one, since G is a self-centred graph of diameter 2. If $|D|=2$, then the claim is true. Suppose $|D| \geq 3$. Let u be a vertex of degree greater than or equal to 2 in <D>. Then all the vertices in D must be adjacent to u.

If not, some vertex $v \in D$ is not adjacent to u. That is v is in the second neighbourhood of u. Since <D> is a connected graph, v must be connected to u through some vertex w in $N_1(u)$. Since u is the internal vertex of <D>, there exists some vertex w', which is adjacent to u in D. Now, to maintain the distance two between v and w', D must

contain some vertex of $N_1(u)$ or $N_2(u)$. In both the cases <D> must have a cycle, which is a contradiction to D is an A.W.C.D set. Hence, all the vertices in D must be adjacent to u. Also, any other vertices, which are different from u, cannot be adjacent. Otherwise, that induces a three cycle in <D>. Hence, $\langle D\rangle \cong K_{1,n}$.
Converse part is a trivial one.

3.2 Bounds on γ_{ac}

Proposition 3.4:

If T is a tree then $\gamma_{ac}(T)$ = p-e, where e is the number of pendant vertices of T.

Proposition 3.5:

$\gamma_{ac}(K_{m,n}) = 2$, for m, n $\geq$ 2.

Proposition 3.6:

$\gamma_{ac}(K_p) = 1$.

Proposition 3.7:

$\gamma_{wc} \leq \gamma_{ac}$, for any graph G, if γ_{ac} exists in G.

Proof:

Any acyclic W.C.D set is a W.C.D set. Therefore, cardinality of minimal W.C.D set is always less than or equal to cardinality of minimal A.W.C.D set. Thus, $\gamma_{wc} \leq \gamma_{ac}$.

Proposition 3.8:

If T is any tree, then $\gamma_{ac} \leq$ p-2.

Proof:

Proof follows from proposition 3.4 and the fact that the number of pendant vertices in a tree is greater than or equal to 2.

Proposition 3.9:

If G is a graph in which γ_{ac} exists, then $\gamma_{ac} \leq$ p-2.

Proof :

Proof follows from the previous proposition 3.8.

Proposition 3.10:

For any isolate-free disconnected graph G, $\gamma_{ac}(\overline{G})=2$.

Proposition 3.11:

Let G be a self-centred graph of diameter 2. If for a vertex u, there exists some adjacent vertices $w, v \in N_2(u)$ such that $N_1(w) \cap N_1(v) = \phi$, then $\gamma_{ac}(\overline{G}) \leq 3$.

Proof :

Let G be a self-centred graph of diameter 2. Suppose for a vertex u, there exists an adjacent vertices $w, v \in N_2(u)$ such that $N_1(w) \cap N_1(v) = \phi$. Then clearly the set $\{u, v, w\}$ will form an A.W.C.D set for $\overline{G}$ and hence $\gamma_{ac}(\overline{G}) \leq 3$.

Proposition 3.12:

For any self-centred graph G of diameter 2, $\gamma_{ac}(G) \leq \Delta+1$.

Proof :

Proof follows directly from the previous proposition 3.3.

Proposition 3.13:

Let G be a self-centred graph of diameter 2 with one vertex of degree 2, then $\gamma_{ac}(G) \leq 3$.

Proof:

Let G be a self-centred graph of diameter 2 with a vertex u' of degree 2. Consider a vertex u, which is adjacent to u'. Consider the other vertex u'', which is adjacent to u'.

Case 1:

If u'' is adjacent to u, that is $u' \in N_1(u)$. Then clearly, all the vertices of $N_2(u)$ is adjacent to u'' (to maintain the distance 2 between them and u'). Also, all of $N_1(u)$ is adjacent to u. Thus, the set $\{u, u''\}$ forms an A.W.C.D. set for G. Hence, $\gamma_{ac} = 2$.

Case 2:

If u'' is not adjacent to u, that is $u'' \in N_2(u)$. To maintain the distance between u' and the vertices of $N_2(u)$, all the vertices of $N_2(u)$ must be adjacent to u''. Also, all the vertices of $N_1(u)$ are adjacent to u. Thus, the set $\{u, u', u''\}$ forms an A.W.C.D set for G. Hence, $\gamma_{ac}(G) \leq 3$.

Therefore, by combining both the cases, we have $\gamma_{ac} \leq 3$.

Proposition 3.14:

Let G be self-centered graph of diameter 2. If G has two non-adjacent vertices of degree 2, then $\gamma_{ac}(G) \leq 3$.

Proof:

Let G be a self-centered graph of diameter 2. Let v_1, v_2 be two non-adjacent vertices of degree 2. Since the graph is of diameter 2, v_1 and v_2 are adjacent to some vertex, say v. Also v_1 and v_2 are of degree 2, they must be adjacent to the vertices say v_1^1 and v_2^1 respectively.

Case 1: If $v_1^1 = v_2^1$

Case 1.1: If v_1^1 lies in $N_1(v)$.

Then all the vertices in $N_2(v)$ are adjacent to v_1^1 to maintain the distance 2 between them and v_1^1. Therefore, v and v_1^1 will form an acyclic W.C.D set for G. Thus $\gamma_{ac}(G) = 2$.

Case 1.2: If $v_1^1 \in N_2(v)$.

In this case, all the vertices of $N_2(v)$ are adjacent to v_1^1. Therefore $\{v, v_1, v_1^1\}$ will form an acyclic W.C.D set for G. Hence $\gamma_{ac}(G) \leq 3$.

Case 2: If $v_1^1 \neq v_2^1$.

Case 2.1: If both v_1^1 and v_2^1 belongs to $N_1(v)$.

Here also to maintain the distance all the vertices of $N_2(v)$ are adjacent to both v_1^1 and v_2^1. Thus $\{v, v_1^1\}$ from and A.W.C.D set for G. Hence $\gamma_{ac}(G) = 2$.

Case 2.2: If suppose $v_1^1 \in N_1(v)$ and $v_2^1 \in N_2(v)$.

Clearly all the vertices of $N_2(v)$ are adjacent to both v_1^1 and v_2^1. Hence $\{v, v_1^1\}$ from A.W.C.D set. Thus $\gamma_{ac}(G) = 2$.

Case 2.3: If both v_1^1 , $v_2^1 \in N_2(v)$.

Then all vertices of $N_2(v)$ are adjacent to both v_1^1 and v_2^1. Therefore, $\{v, v_1, v_1^1\}$ will form an A.W.C.D set. Hence $\gamma_{ac}(G) \leq 3$.

Therefore, combining all the above cases, we get $\gamma_{ac}(G) \leq 3$. Hence the Proof.

Theorem 3.2:

Let G be a self-centered graph of diameter 2. If G has any two adjacent vertices of degree 2, then $\gamma_{ac}(G) \leq 3$.

Proof:

Let G be a self-centered graph of diameter 2. Let u and v be any two adjacent vertices of degree 2. Let $w \neq v$ be a vertex adjacent to u.

Claim 1 : w is not adjacent to v.

If not, let v and w be adjacent in G. Then all the vertices other than u, v and w must be adjacent to w to maintain the distance 2 from u and v. This implies that radius of G is 1. This is a contradiction to G is self-centered of diameter 2. Thus, v and w are non-adjacent.

Therefore, we have, $u \in N_1(w)$ and $v \in N_2(w)$.

Case 1 :

If v is adjacent to some vertex $x \in N_1(w)$ other than u. Then all other vertices in $N_1(w)$ must be adjacent to x to maintain distance 2 from v and also adjacent to w. Clearly in this case, $N_2(w) = \{v\}$ only. Otherwise the distance from u to any other vertex in $N_2(w)$ different from v becomes 3. Thus the edge $\{w, x\}$ forms an A.W.C.D. set for G. Hence $\gamma_{ac}(G) = 2$.

Case 2 :

If v is not adjacent to any of the vertex $N_1(w)$. Clearly $|N_2(w)| = 2$. Let $x \in N_2(w)$ other than v. Now all the vertices of $N_1(w)$ other than u must be adjacent to x to maintain the distance between them and v. Thus the set $\{w, y, x\}$ forms an A.W.C.D set for G. Hence $\gamma_{ac}(G) = 3$.

3.3 Nordhaus-Gaddum Type Results

Next we prove a result relating Nordhaus-Gaddum type result.

Theorem 3.3:

Let G be a graph with diameter greater than or equal to 3, then $\gamma_{ac}(G)+\gamma_{ac}(\overline{G}) \leq p$.

Proof :

Let G be a graph of diameter greater than or equal to 3. Then any pair of vertices in G of distance greater than or equal to 3 will dominate the whole of $\overline{G}$. Hence, $\gamma_{ac}(\overline{G}) \leq 2$. Thus, from the proposition 3.9, we have $\gamma_{ac}(G) + \gamma_{ac}(\overline{G}) \leq p-2+2 = p$.

Theorem 3.4:

If both G and $\overline{G}$ are self-centred graph of diameter 2, then $\gamma_{ac}(G) + \gamma_{ac}(\overline{G}) \leq p+\Delta- \delta + 1$.

Proof :

$$\text{As } \Delta(\overline{G}) = p-\delta-1,\ \gamma_{ac}(G) + \gamma_{ac}(\overline{G}) \leq \Delta(G)+1+ \Delta(\overline{G})+1 = p+ \Delta-\delta+1 .$$

Theorem 3.5:

Let G and $\overline{G}$ be self-centred graph of diameter 2 and let v be a vertex in G. If $N_2(v)$ is a clique, then $\gamma_{ac}(G) + \gamma_{ac}(\overline{G}) \leq \Delta+4$.

Proof :

If $N_2(v)$ is a clique, then take a vertex w in $N_2(v)$ and some $u \in N_1(v)$, which is adjacent to both v and w. Then $\{u, v, w\}$ forms a A.W.C.D set for G. In the complement of G $\gamma_{ac}(\overline{G}) \leq \Delta+1$. Therefore, $\gamma_{ac}(G) + \gamma_{ac}(\overline{G}) \leq \Delta+4$.

Theorem 3.6:

Let v be a vertex in self-centred graph G of diameter 2. If $N_1(v)$ forms an independent set, then $\gamma_{ac}(G) + \gamma_{ac}(\overline{G}) \leq \Delta+4$.

Proof :

The proof follows from the above proof by replacing the role of G and $\overline{G}$.

Theorem 3.7:

Let G and $\overline{G}$ be self-centred graphs of diameter 2. If there exists a vertex v with radius one in γ_{ac} sets for both G and $\overline{G}$, then $\gamma_{ac}(G) + \gamma_{ac}(\overline{G}) \leq p+1$.

Proof:

$$\gamma_{ac}(G) + \gamma_{ac}(\overline{G}) \leq \Delta(G) + 1 + \delta(\overline{G}) + 1 = (p-1) + 2 = p+1.$$

Theorem 3.8:

For any isolate-free disconnected graph G with k components, $\gamma_{ac}(G) + \gamma_{ac}(\overline{G}) \leq p-2(k-1)$.

Proof:

Let $C_1, C_2,\ldots, C_k$ be the k components of G. Then from the proposition 3.5 we get,

$$\begin{aligned}\gamma_{ac}(G) &\leq |C_1|-2 + |C_2|-2 +\ldots+ |C_k|-2\\ &= |C_1|+|C_2|+\ldots+|C_k|-2k\\ &= p-2k\end{aligned}$$

Also, any two vertices one from each of the two different components will form a A.W.C.D set for $\overline{G}$. Thus, $\gamma_{ac}(G) + \gamma_{ac}(\overline{G}) \leq p-2k+2 = p-2(k-1)$.
Hence the proof.

Remark 3.1:
Let r, d be the radius and diameter of G. Let r' and d' be the radius and diameter of <D>, where D is an A.W.C.D set of G.
Then following are the various cases.

As <D> is a tree, we have two major cases (1) d' = 2r' and (2) d' = 2r'-1.

As we have the properties that d-2 $\leq$ d' $\leq$ d and r-1 $\leq$ r' $\leq$ r, let us derive the relation between r and d of G for the various relations between r' and d' in <D>.

Case 1: d' = d and r' = r.

Case 1.1: d' = 2r'

As d = d' = 2r' = 2r, we have d= 2r in this case.

Case 1.2: d' = 2r'-1.

As d = d' = 2r'-1 = 2r-1, we have d = 2r-1 in this case.

Case 2: d = d' and r' = r-1.

Case 2.1: d' = 2r'.

Here, d = d' = 2r' = 2(r-1) = 2r-2. Therefore, d = 2r-2.

Case 2.2: d' = 2r'-1.

Here, d = d' = 2r'-1 = 2(r-1)-1 = 2r-3. Therefore d = 2r-3.

Case 3: d' = d-1 and r' = r.

Case 3.1: d' = 2r'.

Here, d= d'+1 = 2r'+1 = 2r+1, which is not possible in G. Therefore, <D> cannot have this structure.

Case 3.2: d' = 2r'-1.

Here, d = d'+1 = 2r'-1+1=2r. Therefore, in this case d = 2r.

Case 4: d' = d-2 and r' =r.

Case 4.1: d' = 2r'.

Here, d = d'+2 = 2r'+2 = 2r+2, which is not possible in G.

Case 4.2: d' = 2r'-1.

Here, d = d' +2 = 2r'-1+2 = 2r'+1 = 2r+1, which is also not possible in G.

Thus, case 4 is not possible in G.

Case 5: d' = d-2 and r' = r-1.

Case 5.1: d' = 2r'.

Here, d = d' + 2 = 2r'+2 = 2(r-1)+2 =2r. Thus, in this case d = 2r.

Case 5.2: d' = 2r'-1.

Here, d = d'+2 = (2r'-1)+2 = 2r' = 2(r-1) = 2r-2. Thus, d = 2r-2.

Case 6: d' = d-1 and r' = r-1.

Case 6.1: d' = 2r'.

Here, d = d' +1 = 2r'+1 = 2(r-1)+1 =2r-1. Thus, d = 2r-1.

Case 6.2: d' = 2r'-1.

Here, d = d'+1 = (2r'-1)+1 = 2r' = 2(r-1) = 2r-2. Thus, d = 2r-2.

From the above all cases, we have

$2r-3 \leq d \leq 2r$ for a graph G having acyclic dominating set.

Thus we have the following theorem,

Theorem 3.9:

Let G be a connected graph with diameter d and radius r having a γ_{ac} set then $2r-3 \leq d \leq 2r$.

From these above cases it is clear that

1. If G is self-centered with diameter equal to 3 then A.W.C.D set D has d' = 3 and r' =2.

2. If G has diameter 3 and radius 2, then A.W.C.D set D has pair (d', r') with solutions (3, 2) and (1, 1).

3. If G has diameter 4 and radius 2, then (d', r') will be (3, 2), (4, 2), (2, 1).

4. If G has diameter 4 and radius 3, then (d', r') will be (4, 2).

We mention, if G has no A.W.C.D set, then γ_{ac} =0. Then the following results can be easily verified from the solution set (d', r') from (d, r) satisfying $d-2 \leq d' \leq d$ and $r-1 \leq r' \leq r$.

5. If G is self-centered with diameter $d \geq 4$, then G has no A.W.C.D set , that is $\gamma_{ac} = 0$.

6. If G has diameter $d \geq 2r-4$, then $\gamma_{ac} = 0$.

7. If G is self-centered with diameter greater than or equal to 4, then $d_{ac}(G)+d_{ac}(\overline{G}) \leq p/2$.

8. If G is self-centered with diameter 3, then $d_{ac}(G)+d_{ac}(\overline{G}) \leq \frac{p}{4}+\frac{p}{2}=\frac{3p}{4}$, as each dominating set is with $|V(P_4)|=4$.

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Eccentric Domination in Graphs

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Abstract: *A subset D of the vertex set V(G) of a graph G is said to be a dominating set if every vertex not in D is adjacent to at least one vertex in D. A dominating set D is said to be an eccentric dominating set if for every $v \in V-D$, there exists at least one eccentric point of v in D. The minimum of the cardinalities of the eccentric dominating sets of G is called the eccentric domination number $\gamma_{ed}(G)$ of G. In this paper, bounds for γ_{ed}, its exact value for some particular classes of graphs are found.*

Key words: *Eccentric dominating set, Eccentric dominating number.*

1.Introduction

Let G be a finite, simple, undirected (p, q) graph with vertex set V(G) and edge set E(G). For graph theoretic terminology refer to Harary [4], Buckley and Harary [1].

Definition 1.1 Let G be a connected graph and u be a vertex of G. The **eccentricity** e(v) of v is the distance to a vertex farthest from v. Thus, e(v) = max {d(u, v) : u $\in$ V}. The **radius** r(G) is the minimum eccentricity of the vertices, whereas the **diameter** diam(G) is the maximum eccentricity. For any connected graph G, r(G) $\leq$ diam(G) $\leq$ 2r(G). v is a central vertex if e(v) = r(G). The **center C(G)** is the set of all central vertices. The central subgraph < C(G) > of a graph G is the subgraph induced by the center. v is a peripheral vertex if e(v) = d(G). The periphery P(G) is the set of all peripheral vertices.

For a vertex v, each vertex at a distance e(v) from v is an **eccentric vertex**. **Eccentric set of a vertex v** is defined as E(v) = {u $\in$ V(G) / d(u, v) = e(v)}.

Definition 1.2 The **open neighborhood** N(u) of a vertex v is the set of all vertices adjacent to v in G. N[v] = N(v) $\cup$ {v} is called the **closed neighborhood** of v. For a vertex v $\in$ V(G), $N_i(u)$ = {u $\in$ V(G) : d(u, v) = i} is defined to be the **ith neighborhood** of v in G.

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Definition 1.3 Let G be a graph with at least one edge. The set of vertices of **line graph** of G denoted L(G) consists of the edges of G with two vertices of L(G) adjacent whenever the corresponding edges of G are adjacent. A graph G is a **line graph**, if it is isomorphic to the line graph L(H) of some graph H.

Definition 1.4 [6] A set $S \subseteq V$ is said to be a **dominating set** in G, if every vertex in $V-S$ is adjacent to some vertex in S. A dominating set D is an **independent dominating set**, if no two vertices in D are adjacent that is D is an independent set. A dominating set D is a **connected dominating set**, if $< D >$ is a connected subgraph of G. A set $D \subseteq V(G)$ is a **global dominating set**, if D is a dominating set in G and $\overline{G}$.

Definition 1.5 A partition of V(G) is called domatic if all of its classes are dominating sets in G. The maximum number of classes of an domatic partition of V(G) is called the **domatic number** of G and is denoted by $d_d(G)$.

The various domination parameters introduced till now find many applications in covering of entire graph by the different sets with each of which has some specified property. These concepts are helpful to find centrally located sets to cover entire graph in which they are defined. The concept of eccentric set of a node has application in the location of farthest set of a node of a graph and hence in this paper, we define new concept named eccentric domination and study the structural properties of graph using this concept.

2. Eccentric domination

In this we initiate the study of new domination, defined as below.

Definition: 2.1 A set $D \subseteq V(G)$ is an **eccentric dominating set** if D is a dominating set of G and for every $v \in V - D$, there exists at least one eccentric point of v in D.
If D is an eccentric dominating set, then every superset $D' \supseteq D$ is also an eccentric dominating set. But $D'' \subseteq D$ is not necessarily an eccentric dominating set.
An eccentric dominating set D is a **minimal eccentric dominating set** if no proper subset $D'' \subseteq D$ is an eccentric dominating set.

Definition: 2.2 The **eccentric domination number** $\gamma_{ed}(G)$ of a graph G equals the minimum cardinality of an eccentric dominating set. That is, $\gamma_{ed}(G) = \min |D|$, where the minimum is taken over D in D, where D is the set of all minimal eccentric dominating sets of G.

Obviously, $\gamma(G) \leq \gamma_{ed}(G)$.

In the following, we first characterize minimum eccentric dominating set of a graph.

Theorem:2.1

An eccentric dominating set D is a minimal eccentric dominating set if and only if for each vertex u $\in$ D, one of the following is true.

(i) u is an isolated vertex of D or u has no eccentric vertex in D.

(ii) There exists some u $\in$ V–D such that N(u) $\cap$ D = {u}

Proof:

Assume that D is a minimal eccentric dominating set of G. Then for every vertex u $\in$ D, D – {u} is not an eccentric dominating set. That is there exists some vertex v in (V – D) $\cup$ {u} which is not dominated by any vertex in D – {u} or there exists v in (V – D) $\cup$ {u} such that v has no eccentric point in D – {u}.

Case (i):

Suppose u = v , then u is an isolate of D or u has no eccentric vertex in D.

Case (ii):

Suppose v $\in$ V – D

(a) If v is not dominated by D – {u}, but is dominated by D, then v is adjacent to only u in D, that is N(v) $\cap$ D = {u}.

(b) Suppose v has no eccentric point in D – {u} but v has an eccentric point in D. Then u is the only eccentric point of v in D. that is E(v) $\cap$ D = {u}.

Conversely, suppose that D is an eccentric dominating set and for each u $\in$ D one of the conditions holds, we show that D is a minimal eccentric dominating set.

Suppose that D is not a minimal eccentric dominating set, that is, there exists a vertex u $\in$ D such that D – {u} is an eccentric dominating set. Hence, u is adjacent to at least one vertex v in D – {u} and u has an eccentric point in D – {u}.

Therefore, condition (i) does not hold.

Also, if D – {u} is an eccentric dominating set, every element x in V - D is adjacent to at least one vertex in D – {u} and x has an eccentric point in D – {u}.

Hence, condition (ii) does not hold. This is a contradiction to our assumption that for each u $\in$ D, one of the conditions holds. This proves the theorem.

Next, we define eccentric point set of the Graph G and eccentric number of G and establish the relation between the domination number, eccentric number and eccentric domination number of a graph.

Definition: 2.3 Eccentric point set of G:

Let $S \subseteq V(G)$. Then S is known as an eccentric point set of G if for every $v \in V-S$, S has at least one vertex u such that $u \in E(v)$.

An eccentric point set S of G is a minimal eccentric point set if no proper subset S' of S is an eccentric point set of G.

S is known as a minimum eccentric point set if S is an eccentric point set with minimum cardinality.

Let e(G) be the cardinality of a minimum eccentric point set of G. e(G) can be called as **eccentric number of G.**

Let D be a minimum dominating set of a graph G and S be a minimum eccentric point set of G. Clearly, $D \cup S$ is an eccentric dominating set of G. Hence, $\gamma(G)+e(G) \leq \gamma_{ed}(G) \leq (n/2) + e(G)$.

Note: This lower bound is sharp since for the tree $T = K_n + K_1 + K_1 + K_m$, $n, m \geq 2$, $\gamma_{ed}(T) = \gamma(T)+2$, where $e(T) = 2$ for any tree with radius ≥ 2.

The following observations are obvious.

Observation: 1 For any tree T with $|V(T)| \geq 3$, $\gamma_{ed}(T) \leq n- \Delta(T)+2$.

Observation: 2 If G is disconnected, $\gamma(G) = \gamma_{ed}(G)$ since vertices from different components are eccentric to each other.

Observation: 3 $1 \leq \gamma_{ed}(G) \leq n$.

The bounds are sharp, since $\gamma_{ed}(G) = 1$ if and only if $G = K_n$ and $\gamma_{ed}(G) = n$ if and only if $G = \overline{K_n}$.

Observation: 4 $\gamma_{ed}(K_{,n}) = 1$.

The eccentric domination number of some standard classes of graphs are given in the following theorem.

Theorem: 2.2

(i) $\gamma_{ed}(K_{1,n}) = 2$, $n \geq 2$.

(ii) $\gamma_{ed}(K_{m,n}) = 2$.

(iii) $\gamma_{ed}(W_3) = 1$, $\gamma_{ed}(W_4) = 2$, $\gamma_{ed}(W_n) = 3$, for $n = 5$, $\gamma_{ed}(W_6) = 2$, $\gamma_{ed}(W_n) = 3$ for $n \geq 7$.

Proof:

(i) When $G = K_n$, radius = diameter = 1.

Hence any vertex $u \in V(G)$ dominate other vertices and is also an eccentric point of other vertices. Hence, $\gamma_{ed}(K_n) = 1$.

(ii) $G = K_{1,n}$. Let $D = \{u, v\}$, v-central vertex. The central vertex dominate all vertices in $V - D$ and u is an eccentric point of vertices of $V - D$. Hence, $\gamma_{ed}(K_{1,n}) = 2, n \geq 2$

(iii) $G = K_{m,n}$. $V(G) = V_1 \cup V_2$. $|V_1| = m$ and $|V_2| = n$ such that each element of V_1 is adjacent to every vertex of V_2 and vice versa.

Let $D = \{u, v\}$, $u \in V_1$ and $v \in V_2$. u dominate all the vertices of V_2 and it is eccentric to elements of $V_1 - \{u\}$. Similarly, v dominates all the vertices of V_1 and it is eccentric to elements of $V_2 - \{v\}$. Hence, D is a minimum eccentric dominating set and hence $\gamma_{ed}(K_{m,n}) = 2$.

(iv) $G = W_3 = K_4$. Hence, $\gamma_{ed}(W_3) = 1$

When $G = W_4$, Consider $D = \{u, v\}$, where u and v are adjacent non central vertices. D is a minimum eccentric dominating set. Therefore, $\gamma_{ed}(W_4) = 2$.

When $G = W_n$, let $D = \{u, v, w\}$ where u and v are any two adjacent non central vertices and w is the central vertex.. Then D is a minimum eccentric dominating set of G. Therefore, $\gamma_{ed}(W_n) = 3$, $n \geq 5$.

The next theorem gives exact value for eccentric domination number of graph obtained from deletion of a perfect matching (linear factor) from a complete graph on even number of vertices.

Theorem: 2.3

Let n be an even integer. Let G be obtained from the complete graph K_n by deleting edges of a linear factor. Then $\gamma_{ed}(G) = n/2$.

Proof:

Let u and v be a pair of non adjacent vertices in G. Then u and v are eccentric to each other. Also, G is unique eccentric point graph. Therefore, $\gamma_{ed}(G) \geq n/2$.

Consider $D \subseteq V(G)$ such that $\langle D \rangle = K_{n/2}$. D contains n/2 vertices such that each vertex in V–D is adjacent to at least one element in D and each element in V–D has its eccentric point in D. Hence $\gamma_{ed}(G) \leq n/2$. From (1) and (2) $\gamma_{ed}(G) = n/2$.

Following theorems give upper bound for eccentric domination number of a graph.

Theorem: 2.4

If G is of radius one and diameter two, then $\gamma_{ed}(G) \leq (n-t+2)/2$ where t is the number of vertices with eccentricity one.

Proof: Let $u \in V(G)$ such that $e(u) = 1$. Let t be the number of vertices with eccentricity one. u dominates all other vertices and for t–1 other vertices u is an eccentric point. Consider the remaining (n–t) vertices of G. They are also dominated by u but their eccentric points are different from u.

Hence, $\gamma_{ed}(G) \leq 1+(n-t)/2 = (n-t+2)/2$.

Theorem: 2.5

If G is of diameter two $\gamma_{ed}(G) \leq 1+\delta(G)$.

Proof: $diam(G) = 2$. Let $u \in V(G)$ such that $deg_G u = \delta(G)$. Consider, $D = \{u\} \cup N(u)$. This is an eccentric dominating set for G. Therefore, $\gamma_{ed}(G) \leq \delta(G)+1$ and D is a connected eccentric dominating set.

Corollary: 2.5

If G is self centered of diameter 2, then $\gamma_{ed}(G) + \gamma_{ed}(\overline{G}) \leq n + \delta - \Delta + 1$.

Proof: By Theorem 2.5, $\gamma_{ed}(G) \leq 1 + \delta$ and $\gamma_{ed}(\overline{G}) \leq 1+\delta(\overline{G}) = 1 + \overline{\delta}$. Hence $\gamma_{ed}(G) + \gamma_{ed}(\overline{G}) \leq 1 + \delta + 1 + \overline{\delta} = 2 + \delta + (n - 1 - \Delta) = n + \delta - \Delta + 1$.

If there exists no x in $N_2(u)$ such that x is adjacent to all vertices of N(u) and there exists no vertex y in N(u) which is adjacent to all vertices of $N_2(u)$ then N(u) is eccentric dominating set of G and $N_2(u)$ is an dominating set of $\overline{G}$. Thus, $\gamma_{ed}(G) + \gamma_{ed}(\overline{G}) \leq n + \delta - \Delta - 1$.

Theorem: 2.6

If G is of radius two and diameter three, then $\gamma_{ed}(G) \leq \min \{(n + deg_G u - 1)/2\}$, where the minimum is taken over all central vertices.

Proof: Let u be a central vertex with minimum degree. Consider N(u). N(u) dominates all the vertices of G and all the vertices in $N_2(u)$ are eccentric to u. Let S be a subset of $N_2(u)$ with minimum cardinality such that vertices in $N_2(u) - S$ has their eccentric vertices in S. Then $|S| \leq |N_2(u)|/2 = (n - deg_G u - 1)/2$. Now $N(u) \cup S$ is an eccentric dominating set of G. Hence, $\gamma_{ed}(G) \leq deg_G u + (n - deg_G u - 1)/2 = (n + deg_G u - 1)/2$. This proves the theorem.

Theorem: 2.7

If G is of radius two and diameter three, then $\gamma_{ed}(G) \leq \min \{n - deg_G u/2\}$, where the minimum is taken over all central vertices.

Proof: Let u be a central vertex with maximum degree. Then $V - N(u)$ is a dominating set of G. Vertices of N(u) are dominated by u, but vertices of N(u) may have their eccentric

vertices in N(u) also. Let S be a subset of N(u) with minimum cardinality such that vertices in N(u) $-$ S has their eccentric vertices in S. Then $|S| \leq deg_G u\ /2$. Now $(V-N(u)) \cup S$ is an eccentric dominating set of G. Hence, $\gamma_{ed}(G) \leq deg_G u/2 + (n - deg_G u) = n - deg_G u/2$. Hence, $\gamma_{ed}(G) \leq \min\{n - deg_G u/2\}$.

Corollary: 2.7.1

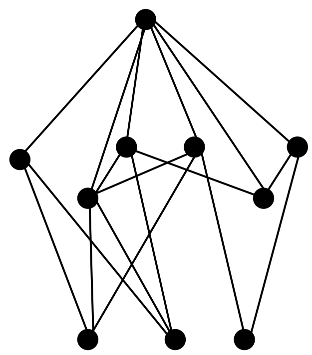
$\gamma_{ed}(G) \leq \min\{n - deg_G u/2, (n + deg_G u-1)/2\}$, where the minimum is taken over all central vertices.

Corollary: 2.7.2

If G is of radius two and diameter three and if G has a pendent vertex v of eccentricity 3 then $\gamma_{ed}(G) \leq \Delta(G)$.

Proof: If G has a pendent vertex v of eccentricity 3 then its support u is of eccentricity 2. In this case N(u) is an eccentric dominating set. Thus, $\gamma_{ed}(G) \leq deg_G u \leq \Delta(G)$.

Example: Consider the following graph G.



By the theorem, $\gamma_{ed}(G) \leq \min\{n - deg_G u/2, (n + deg_G u-1)/2\} = \min\{6,7,8\} = 6$.

Theorem :2.8

If G is of radius 2 with a unique central vertex u then $\gamma_{ed}(G) \leq n - deg(u)$.

Proof: If G is of radius 2 with a unique central vertex u then u dominates N[u] and the vertices in V $-$ N[u] dominate themselves and each vertex in N(u) has eccentric vertices in V $-$ N[u] only. Therefore, V $-$ N(u) is an eccentric dominating set of cardinality $n - deg(u)$, so that $\gamma_{ed}(G) \leq n - deg(u)$.

Corollary: 2.8

(i) If G is a unicentral tree of radius 2, then $\gamma_{ed}(G) \leq n - deg(u)$, where u is the central vertex.

(ii) If G is a bicentral tree of radius 2, then $\gamma_{ed}(G) \leq n - deg(u)+1$, where u is a central vertex.

Theorem: 2.9

If G is of radius greater than two, then $\gamma_{ed}(G) \leq n - \Delta(G)$.

Proof: Let u be a vertex of maximum degree $\Delta(G)$. Then u dominates N[u] and the vertices in V − N[u] dominate themselves. Also, since diam(G) > 2, each vertex in N(u) has eccentric vertices in V − N[u] only. Therefore, V − N(u) is an eccentric dominating set of cardinality $n - \Delta(G)$, so that $\gamma_{ed}(G) \leq n - \Delta(G)$.

In the following three theorems, we analyze the bounds of eccentric domination number of a tree in terms of its domination number.

Theorem: 2.10

For a tree T, $\gamma(T) \leq \gamma_{ed}(T) \leq \gamma(T)+2$.

Proof:

Obviously, $\gamma(T) \leq \gamma_{ed}(T)$. Let d be the diameter of T. Let u, v $\in$ V(T) such that e(u) = e(v) = d and d(u, v) = d. Then for any w $\in$ V(T) either u or v is an eccentric point. Let D be any γ-dominating set of T. Then D $\cup$ {u, v} is an eccentric dominating set of T. Hence, $\gamma_{ed}(T) \leq \gamma(T)+2$.

The next theorem gives an upper bound for eccentric domination number of a tree in terms of its maximum degree.

Theorem:2.11

For a tree T, $\gamma_{ed}(T) \leq n - \Delta(T) +1$.

Proof: If T has a vertex u of maximum degree which is not a support, then V − N(u) is an eccentric dominating set of T. If T has a vertex u of maximum degree which is a support of a pendent vertex v, then V − [N(u) − v] is an eccentric dominating set of T. Hence the theorem follows.

Theorem:2.12

For a tree T with radius greater than two, $\gamma_{ed}(T) < n - \Delta(T)$.

Proof: As in theorem 2.9, V − N(u) is an eccentric dominating set of cardinality $n - \Delta(G)$. Since the radius of T is atleast three, diameter of T is atleast 5. Consider a diametral

path P. This path contains atleast six vertices and includes atmost two edges from the subgraph induced by N[u], that is it contains atmost three vertices from N[u].

Case (i): All vertices of P − N[u] (except pendent vertices) are support of some pendent vertices

In this case, we have to include all the vertices of P in a γ_{ed} set, but we can leave those pendent vertices from V − N(u) to form a γ_{ed} set. Therefore, $\gamma_{ed}(T) < n - \Delta(T)$.

Case (ii): Atleast one vertex of P − N[u] (except pendent vertices) is not a support

In this case, we can leave that vertex from V − N(u) to form a γ_{ed} set. Therefore, $\gamma_{ed}(T) < n - \Delta(T)$.

Hence atleast one vertex can be deleted from V − N(u) to form a minimal eccentric dominating set. Hence $\gamma_{ed}(T) < n - \Delta(T)$.

Theorem: 2.13

Let T be a tree with diameter d > 2. $\gamma(T) = \gamma_{ed}(T)$ if and only if T has a $\gamma(T)$ dominating set D containing at least two (pendent) peripheral vertices at distance d to each other.

Proof:

Assume $\gamma(T) = \gamma_{ed}(T)$. Let D be an eccentric dominating set with cardinality $\gamma(T) = \gamma_{ed}(T)$. Since in a tree, eccentric vertex of any vertex is a peripheral vertex D contains at least two peripheral vertices at distance d to each other.

On the other hand, assume that D is a $\gamma(T)$ dominating set of T containing at least two peripheral vertices u, v at distance d to each other. Every vertex $x \in V(T)$ has either u or v as an eccentric vertex. Hence D is also an eccentric dominating set of T. Hence, $\gamma(T) = \gamma_{ed}(T)$.

Example: (i) $\gamma(P_4) = 2 = \gamma_{ed}(P_4)$

(ii) $\gamma(K_{1,n}) = 1$, $\gamma_{ed}(K_{1,n}) = 2 = \gamma(K_{1,n})+1$

(iii) If $G = K_n + K_1 + K_1 + K_m$, n, m ≥ 2, $\gamma_{ed}(G) = 4 = \gamma(G) + 2$.

Theorem: 2.14 For a bicentral tree T with radius 2, $\gamma_{ed}(T) \leq \min\{n - \Delta(T)+1, 4\}$

Proof: Let u and v be the central vertices of T, Then N[u] and N[v] are eccentric dominating sets of T. V − [N(u) − v], V − [N(v) − u] are also eccentric dominating sets of T. Also deg u + deg v = n. Hence, $\gamma_{ed}(T) \leq n - \Delta(T)+ 1$. All the four vertices of a diametral path also form an eccentric dominating set. Hence the theorem follows.

Corollary:2.14 (i) For a bicentral tree T with radius 2, $\gamma_{ed}(T) = 2$ if and only if $T = P_4$.

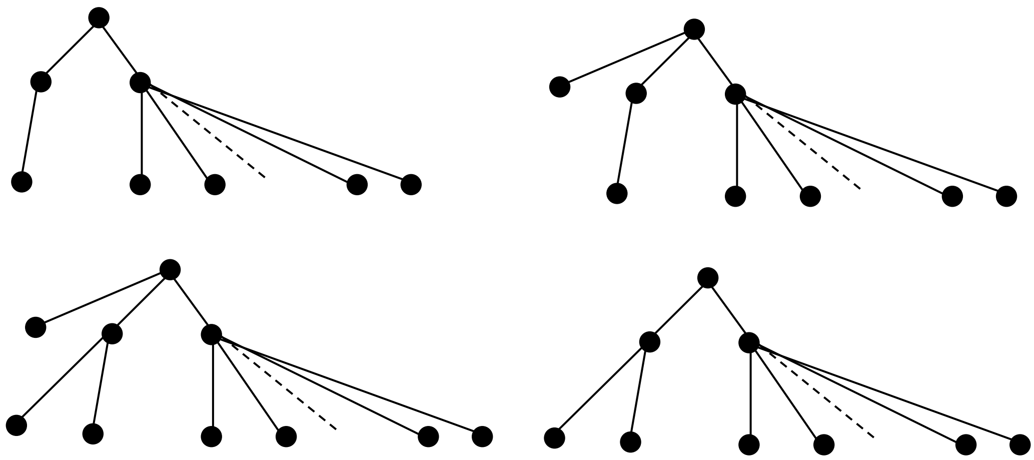
(ii) For a bicentral tree $T \neq P_4$ with radius 2, $\gamma_{ed}(T) = 3 = n - \Delta(T)+1$ if and only if T is a wounded spider having atmost one non wounded leg.

(iii) For a bicentral tree T with radius 2, $\gamma_{ed}(T) = 4$ if and only if degree of the central vertices are ≥ 3.

Theorem: 2.14 If T is a wounded spider having atmost one non wounded leg, $\gamma_{ed}(T) = n - \Delta(T) +1$.

Proof: Proof is obvious.

Theorem: 2.15 Let T be a tree with radius 2 and diameter 4. $\gamma_{ed}(T) = n - \Delta(T)$ if and only if any one of the following is true: (i) $T = P_5$. (ii) T is a wounded spider having at least two non wounded legs. (iii) T is any one of the following four types of trees.



Proof: When T is a wounded spider having at least two non wounded legs or any one of the trees given above, it is clear that $\gamma_{ed}(T) = n - \Delta(T)$.

On the other hand assume that $\gamma_{ed}(T) = n - \Delta(T)$. Since T is a tree with radius 2 and diameter 4, T has a unique centre v.

Case (i): Let deg v = $\Delta(T)$.

Consider V – N(v). N(v) and V – N(v) are independent sets. N(v) is not an eccentric dominating set, since v has no eccentric vertex in N(v). Also each vertex in N(v) has either 0 or 1 neighbor in V – N(v). Hence T is a wounded spider having at least two non wounded legs.

Case (ii): Let deg v $\neq \Delta(T)$.

Let u be a vertex of maximum degree. It cannot be a pendent vertex. Therefore eccentricity of u is three. V – N(u) is an eccentric dominating set implies V – N(u) contains atleast one peripheral vertex (which is eccentric to u).

Now, consider the branches of T taking v as a root. If T contains a third branch with a peripheral vertex then V – N(u) is not a minimum eccentric dominating set. Therefore third branch must not contain any vertex of eccentricity 4. Suppose T contains more than three branches then also V – N(u) is not a minimum eccentric dominating set. Hence degree of v must be 2 or 3, with one branch containing u, second branch containing vertices of eccentricity 4 and the third one having a vertex of eccentricity 3 as end vertex or not. If the second branch contains more than two peripheral vertices then also V – N(u) is not an minimum eccentric dominating set. Hence T must be any one of the given four types of trees.

Next we find the exact value of eccentric domination number of a path.

Remark: 2.1 Let D be an eccentric dominating set of a path. Then any one of the following is true.

(i) D is a dominating set containing at least two peripheral nodes at distanced d to each other. That is D contains end vertices of the path.

(ii) D is a dominating set containing one peripheral vertex v and all vertices lying on the shortest path from v to a central node.

Remark: 2.2

An eccentric dominating set D of a path contains minimum number of vertices only when D contains two peripheral vertices at distance d (= diameter) to each other. That is D contains end vertices of the path.

Theorem: 2.16 $\gamma_{ed}(P_n) = \lceil n/3 \rceil$, if n = 3k+1,

$\gamma_{ed}(P_n) = \lceil n/3 \rceil + 1$, if n = 3k or 3k+2.

Proof: Case (i) n = 3k

An eccentric dominating set of P_n must contain the two end vertices.

Let $v_1, v_2, v_3, v_4, \ldots, v_{3k}$ represent the path P_n. D = $\{v_2, v_5, v_8, \ldots, v_{3k-1}\}$ is the only γ-dominating set of P_n. D is not an eccentric dominating set.

$D' = \{v_1, v_4, v_7, \ldots, v_{3k-2}, v_{3k}\}$ is an eccentric minimum dominating set and $|D'| = k+1 = \gamma(P_n) + 1$. Hence $\gamma_{ed}(P_{3k}) = \gamma(P_{3k})+1 = \lceil n/3 \rceil + 1$.

Case (ii) n = 3k+1

D = $\{v_1, v_4, v_7, \ldots, v_{3k-2}, v_{3k+1}\}$ is the minimum dominating set in P_n. It contains the two end vertices. Hence it is also an eccentric dominating set.

Hence $\gamma_{ed}(P_n) = \gamma(P_n) = \lceil n/3 \rceil$.

Case (iii) n = 3k+2

D = $\{v_2, v_5, v_8, \ldots, v_{3k+2}\}$ is a minimum dominating set. It contains one end vertex v_{3k+2} and it is not an eccentric dominating set. (other minimum dominating sets are also not eccentric). Hence D $\cup$ $\{v_1\}$ is a minimum eccentric dominating set. Therefore, $\gamma_{ed}(P_n)$ = $\gamma(P_n)+1 = \lceil n/3 \rceil +1$.

Following two theorems give the exact value of the eccentric domination number of cycles and their complement graphs.

Theorem: 2.17 (i) $\gamma_{ed}(C_n) = n/2$ if n is even.

(ii) $\gamma_{ed}(C_n) = \lceil n/3 \rceil$ or $\lceil n/3 \rceil +1$, if n is odd.

Proof of (i):

If n = 4, any two adjacent vertices of C_4 is an eccentric dominating set of C_4.

Hence $\gamma_{ed}(C_4) = 2$.

Let n = 2k and k > 2.

Let the cycle C_n be $v_1\, v_2\, v_3 \ldots v_{2k} v_1$. Each vertex of C_n has exactly one eccentric vertex (that is C_n is unique eccentric point graph).

Hence $\gamma_{ed}(C_n) \geq n/2$. ------------------- (1)

case(i) k-odd.

Consider D = $\{v_1, v_3, \ldots, v_k, v_{k+2}, \ldots, v_{2k-1}\}$. This D is an eccentric dominating set for C_n since D dominates C_n and v_i is an eccentric point of v_{i+k}.

Hence $\gamma_{ed}(C_n) \leq n/2$. ----------------------------------- (2)

From (1) and (2) $\gamma_{ed}(C_n) = n/2$.

case(ii) k even.

Let D = $\{v_1, v_3, \ldots, v_{k-1}, v_{k+2}, \ldots, v_{2k}\}$. This D is an eccentric dominating set for C_n, since D dominates C_n and v_i is an eccentric point of v_{i+k}.

Hence $\gamma_{ed}(C_n) \leq n/2$. ------------------------------- (3)

From (1) and (3) $\gamma_{ed}(C_n) = n/2$.

Proof of (ii):

When n is odd, each vertex of C_n has exactly two eccentric vertices.

If n = 2k+1, $v_i \in V(G)$ has v_{i+k}, v_{i+k+1} as eccentric points.

case(i) n = 3m, n odd $\Longrightarrow$ m odd

$$n = 3m = 2k+1 \Longrightarrow 2k \text{ even and } 2k = 3m-1$$

$$2k = 3(m-1)+2$$

$$k = (3(m-1)+2)/2 \Longrightarrow k = 3l+1 \text{ (since } m-1 \text{ is even)}$$

Consider $D = \{v_1, v_4, v_7, \ldots, v_k, v_{k+3}, \ldots, v_{2k-1}\}$. D is an eccentric dominating set and D is a γ-dominating set of C_n and $|D| = n/3 = m$

$$\text{Hence, } \gamma_{ed}(C_n) = n/3 = \gamma(C_n).$$

case(ii) $n = 3m+1$, n odd $\Rightarrow$ m is even.

Also $3m = 2k \Rightarrow$ k is a multiple of 3.

Consider $D = \{v_1, v_4, \ldots, v_{k+1}, v_{k+3}, v_{k+6}, \ldots, v_{2k-1}\}$. D is an eccentric dominating set and $|D| = \lceil n/3 \rceil = \gamma(C_n)$. Hence $\gamma_{ed}(C_n) = \lceil n/3 \rceil = m+1$.

case(ii) $n = 3m+2 \Rightarrow 3m$ is odd $\Rightarrow$ m is odd.

$$2k = 3m+1 = 3(m-1) + 4$$
$$k = 3l + 2$$

Consider $D = \{v_1, v_4, \ldots, v_{k-1}, v_k, v_{k+3}, \ldots, v_{2k+1}\}$. D is an eccentric dominating set with $\lceil n/3 \rceil + 1$ vertices and no γ-dominating set of C_n is an eccentric dominating set of C_n.

$$\text{Hence } \gamma_{ed}(C_n) = \lceil n/3 \rceil + 1.$$

Theorem: 2.18 $\gamma_{ed}(\overline{C_4}) = 2$, $\gamma_{ed}(\overline{C_5}) = 3$ and $\gamma_{ed}(\overline{C_n}) = \lceil n/3 \rceil$, $n \geq 6$.

Proof: Clearly, $\gamma_{ed}(\overline{C_4}) = 2$, $\gamma_{ed}(\overline{C_5}) = 3$, by Observation 2 and by Theorem 2.15.

Now, assume that $n \geq 6$. Let $v_1, v_2, v_3, \ldots, v_n, v_1$ form C_n. Then $\overline{C_n} = K_n - C_n$ and each vertex v_i is adjacent to all other vertices except v_{i-1}, and v_{i+1} in $\overline{C_n}$. Hence eccentric point of v_i in $\overline{C_n}$ is v_{i-1}, and v_{i+1} only. Hence any eccentric dominating set must contain either v_i or any one of v_{i-1}, v_{i+1}. So, $\gamma_{ed}(\overline{C_n}) \geq \lceil n/3 \rceil$, Now, we can consider an eccentric (minimal) dominating set as follows.

$\{v_1, v_4, v_7, \ldots, v_{3m-2}\}$ if $n = 3m$;

$\{v_1, v_4, v_7, \ldots, v_{3m+1}\}$ if $n = 3m+1$;

$\{v_1, v_4, v_7, \ldots, v_{3m+1}, v_{3m+2}\}$ if $n = 3m+2$;

Hence $\gamma_{ed}(\overline{C_n}) \leq \lceil n/3 \rceil$. Thus, it follows that $\gamma_{ed}(\overline{C_n}) = \lceil n/3 \rceil$, for $n \geq 6$.

Parthasarathy and Nandakumar introduced the concept of eccentricity preserving spanning trees of a given graph and its structural properties studied by them and also by Janakiraman [5]. Next theorem gives bound for the introduced parameter for the graphs having eccentricity preserving spanning trees.

Theorem: 2.19 If G has an eccentricity preserving spanning tree then $\gamma_{ed}(G) \leq \gamma(G) + 2$.

Proof: If G has an eccentricity preserving spanning tree, then the minimum number of vertices which are eccentric points of other vertices are 2.

$$\text{Hence, } \gamma_{ed}(G) \leq \gamma(G) + 2.$$

Next we give two results relating global domination number and eccentric domination number of a graph.

Observation: 2.4 If $\gamma(G) \geq 3$, γ_{ed} dominating set D of G is also a dominating set of $\overline{G}$. Hence $\gamma_g(G) \leq \gamma_{ed}(G)$.

Theorem: 2.20 If G is self-centered of diameter two, then $\gamma_g(G) \leq \gamma_{ed}(G)$.

Proof: Let D be an eccentric dominating set of G. If v $\in$ V–D, there exist u, w $\in$ D such that uv $\in$ E(G) and w is eccentric to v. In $\overline{G}$, v and w are adjacent. So again, D is a global dominating set. Hence $\gamma_g \leq \gamma_{ed}$.

Remark: 2.3 Minimum eccentric dominating set need not be a minimum global dominating set.

In $\overline{K_2}+K_1+K_1+\overline{K_2}$, $\gamma_{ed} = 4$ and $\gamma_g = 3$..

Following result gives the exact bound for a product graph of a graph.

Theorem: 2.21 Let G be a connected graph with $|V(G)| = n$. Then $\gamma_{ed}(G \circ K_1) = n$.

Proof: Let $V(G) = \{v_1, v_2, \ldots, v_n\}$. Let v_i' be the pendent vertex adjacent to v_i in $G \circ K_1$ for i = 1, 2, …, n.

Then $\{v_1', v_2', \ldots, v_n'\}$ is an eccentric dominating set for $G \circ K_1$, and is also a minimum dominating set for $G \circ K_1$. Hence $\gamma_{ed}(G \circ K_1) = n$.

Next, we characterize some special classes of graph for which eccentric domination number is 1 or 2.

Theorem: 2.22 $\gamma_{ed}(G) = 1$ if and only if $G = K_n$.

Proof: If $G \neq K_n$, then G has atleast one pair of non-adjacent vertices with eccentricity greater than one.

Theorem: 2.23 Let G be a connected graph. Then $\gamma_{ed}(G) = 2$ if and only if G is any one of the following.

(i) r(G) = 1, d(G) = 2 and u $\in$ V(G) such that e(u) = 2 and d(u, v) = 2 for all v $\in$ V(G) with e(v) = 2.

(ii) G is self-centered of diameter 2, having a dominating edge which is not in a triangle.

(iii) r(G) = 2, d(G) = 3 and G has a γ-set D of cardinality two which is not connected.

Proof: When G satisfies any one of the above conditions obviously $\gamma_{ed}(G) = 2$.

On the other hand, assume that $\gamma_{ed}(G) = 2$. Therefore, $\gamma(G) = 1$ or $\gamma(G) = 2$.

Case(i) $\gamma(G) = 1$ and $\gamma_{ed}(G) = 2$. This implies G satisfies (i).

Case(ii) $\gamma(G) = 2 = \gamma_{ed}(G)$

Let D be a minimum γ_{ed}-dominating set of G. Let $D = \{u, v\} \subseteq V(G)$.

Since $\gamma(G) = 2$, $r(G) \geq 2$.

(a) <D> is connected:

Since D is connected u and v are adjacent and the edge uv is a dominating edge for G. Therefore $r(G) \geq 2$ and $2 \leq d(G) \leq 3$. Suppose $d(G) = 3$, there exists a vertex x with eccentricity 3 and x is dominated by u or v.

Let $xu \in E(G)$. Now, D is an γ_{ed}-set. Hence v must be an eccentric point of x. This implies that $d(x, v) = 3$, But x u v is a path $\Longrightarrow d(x, v) = 2$, which is a contradiction. Hence, x must be a vertex with eccentricity 2. This implies that $d(G) = 2$, that is G is self-centered with diameter 2. [There exists no w, adjacent to both u and v, since in that case, w has no eccentric point in D, since $r(G) \geq 2$]

(b) <D>is not connected:

G is a connected graph, $\gamma(G) = 2 = \gamma_{ed}(G)$ implies that $d(G) \leq 3$. Therefore $d(u, v) = 2$ or 3.

If $d(u, v) = 2$, there exists $w \in V(G)$ such that w is adjacent to both u and v. Therefore, w must be of eccentricity one, since $D = \{u, v\}$ is an eccentric dominating set. Thus G is a graph with $r(G) = 1$ and $d(G) = 2$ which is a contradiction to $\gamma(G) = 2$. Hence $d(u, v) \neq 2$. This implies that $d(u, v) = 3$. Then $e(u) = e(v) = 3$ and uwxv is a shortest path, since D is an eccentric dominating set, eccentric point of w must be v and eccentric point of x must be u. Therefore, $e(w) = e(x) = 2$. Thus G is a connected graph with radius 2 and diameter 3 with a γ-set D of cardinality two which is not connected. This proves the theorem.

Corollary 2.23: $\gamma(P_4) = 2$ and $\gamma(K_{1,n}) = 2$.

Theorem: 2.24 $\gamma_{ed}(G) = 2$ and D is an eccentric dominating set $\cong K_2$ if and only if every vertex in V–D is adjacent to exactly one of u and v and G is self-centered with diameter 2.

Proof: Follows from case (ii)(a) of previous theorem.

Theorem: 2.25 If G is a graph with diameter three and $\gamma(G) = 2$ then $\gamma_{ed}(G) = 2$ if and only if G has a pair {u, v} such that one is the unique eccentric point of other and d(u, v) = 3

Proof: Follows from case (ii)(b) of previous theorem.

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Acyclic Weak Convex Domination Critical Graphs

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Abstract: *In a graph $G = (V, E)$, a set $D \subset V$ is a weak convex dominating(WCD) set if each vertex of V-D is adjacent to at least one vertex in D and $d_{<D>}(u,v) = d_G(u,v)$ for any two vertices u, v in D. A weak convex dominating set D, whose induced graph <D> has no cycle is called acyclic weak convex dominating(AWCD) set. The domination number $\gamma_{ac}(G)$ is the smallest order of a acyclic weak convex dominating set of G and the codomination number of G, written $\gamma_{ac}(\bar{G})$, is the acyclic weak convex domination number of its complement. In this paper we study the change in the behaviour of acyclic weak convex domination number with respect to addition of edges in the respective graph.*

Keywords: *domination number, distance, eccentricity, radius, diameter, self-centered, neighbourhood, weak convex dominating set, acyclic weak convex dominating set.*

1. Introduction

Graphs discussed in this paper are undirected and simple. Unless otherwise stated the graphs which we consider are connected graphs only. For a graph G, let V(G) and E(G) denote its vertex and edge set respectively. The *degree* of a vertex *v* in a graph G is denoted by $\deg_G(v)$. The length of any shortest path between any two vertices *u* and *v* of a connected graph G is called the *distance between u and v* and is denoted by $d_G(u, v)$. The distance between two vertices in different components of a disconnected graph is defined to be ∞. For a connected graph G, the *eccentricity* $e_G(v) = \max\{d_G(u, v): u \in V(G)\}$. If there is no confusion, we simply use the notion $\deg(v)$, $d(u, v)$ and $e(v)$ to denote degree, distance and eccentricity respectively for the concerned graph. The minimum and maximum eccentricities are the *radius* and *diameter* of G, denoted r(G) and diam(G) respectively. When these two are equal, the graph is called *self-centered* graph with radius r, equivalently is *r self-centered*. A vertex *u* is said to be an *eccentric vertex* of *v* in a graph G, if $d(u, v) = e(v)$. In general, *u* is called an eccentric vertex, if it is an eccentric vertex of some vertex. For $v \in V(G)$, the *neighbourhood* $N_G(v)$ of *v* is the set of all vertices adjacent to *v* in G. The set $N_G(v) = N_G(v) \cup \{v\}$ is called the *closed neighbourhood* of *v*. A set S of edges in a graph is said to be *independent* if no two of the edges in S are adjacent. An edge

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e = (u, v) is a *dominating edge* in a graph G if every vertex of G is adjacent to at least one of u and v.

The concept of domination in graphs was introduced by Ore. A set D$\subseteq$V(G) is called *dominating set* of G if every vertex in V(G)-D is adjacent to some vertex in D. D is said to be a *minimal* dominating set if D-{v} is not a dominating set for any $v\in$D. The *domination number* γ(G) of G is the minimum cardinality of a dominating set. We call a set of vertices a γ-set if it is a dominating set with cardinality γ(G). Different types of dominating sets have been studied by imposing conditions on the dominating sets. A dominating set D is called *connected* (*independent*) dominating set if the induced subgraph <D> is connected (independent). D is called a *total dominating set* if every vertex in V(G) is adjacent to some vertex in D.

A cycle D of a graph G is called a *dominating cycle* of G, if every vertex in V – D is adjacent to some vertex in D. A dominating set D of a graph G is called a *clique dominating set* of G if <D> is complete. A set D is called an *efficient dominating set* of G if every vertex in V – D is adjacent to exactly one vertex in D. A set D $\subseteq$ V is called a *global dominating set* if D is a dominating set in G and $\overline{G}$. A set D is called a *restrained dominating set* if every vertex in V(G)-D is adjacent to a vertex in D and another vertex in V(G)-D. A set D is a *weak convex dominating set* if each vertex of V-D is adjacent to at least one vertex in D and the distance between any two vertices u and v in the induced graph <D> is equal to that of those vertices u and v in G. By $\gamma_c, \gamma_i, \gamma_t, \gamma_o, \gamma_k, \gamma_e, \gamma_g, \gamma_r$ and γ_{wc}, we mean the minimum cardinality of a connected dominating set, independent dominating set, total dominating set, cycle dominating set, clique dominating set, efficient dominating set, global dominating set and restrained dominating set respectively.

In this paper we study the change in the behaviour of acyclic weak convex domination number with respect to addition of edges in the respective graph. In this paper we define a graph called k – Acyclic Weak Convex Domination critical graph and study the properties possessed by the graph with respect to k=2 and 3, and from which, we generalised some of the properties with respect to any value of k. Also we studied several interesting properties with respect to the diameter and radius of the graph.

2.Main Results

Definition 2.1:

A graph is said to be A.W.C.D critical graph, if for every edge e$\notin$E(G), γ_{ac}(G+e) < γ_{ac}(G).

If G is A.W.C.D critical graph with $\gamma_{ac}(G)$ = k, then G is said to be k – A.W.C.D critical graph.

Observations :

2.1 : A graph G is 1-critical $\Leftrightarrow$ G = K_p.

2.2 : If γ_{ac} = 2, then γ_{wc} = 2.

Theorem 2.1:

G is 2- A.W.C.D critical $\Leftrightarrow$ the following hold good;

(i) G is 2-domination; and

(ii) For any two non-adjacent vertices one of them is of degree (n-2).

Theorem 2.2:

For any 2-A.W.C.D graph, $\Delta \leq$ n - 2 and $\delta \leq$ n – 3.

Theorem 2.3:

Any 2-A.W.C.D critical graph has diameter which equals to two.

Proof:

Let x and y be any two non-adjacent vertices of G. Then in G + xy either {x} or {y} will form a dominating set. Without loss of generality, assume that {x} will form a generating set. Then all the neighbours of $N_1(y)$ will be adjacent with x in G+ xy and hence in G also. Therefore, diameter of G must be equal to 2 as it cannot be equal to one.

Theorem 2.4:

Any 2-A.W.C.D critical graph is a block.

Proof:

Let v be a cut vertex of G and v_1 be the pendent vertex joined with v. Let {u, v} be a dominating edge of G. Then there exists at least one vertex $u_1 \in N_1(u)$ such that $d(u_1,v_1) \geq 3$, contradictory to the above theorem 2.3. Hence G is a block.

Theorem 2.5:

There exists no graph G for which both G and $\overline{G}$ are 2-A.W.C.D critical.

Theorem 2.6:

The diameter of a 3 - A.W.C.D critical graph is at most 3.

Proof:

Let *x* and *y* be any two non-adjacent vertices of G. Now G+*xy* has an A.W.C.D set of cardinality 2. Let it be {*x*, *z*} (or {*y*, *z*}). If *x* dominates some vertex in $N_1(y)$, then

$d(x, y) \leq 2$, otherwise z must dominate all of $N_1(y)$ in G+xy and hence in G also. This implies that $d(x, y) \leq 3$.

Theorem 2.7:

If u is a cut vertex of a 3-A.W.C.D critical graph G, then u is adjacent to a pendant vertex of G.

Corollary 2.1:

If G is a 3-A.W.C.D graph with $\delta \geq 2$, then G is a block.

Theorem 2.8:

Let G be a 3-A.W.C.D critical graph, then there cannot be any geodesic cycle of length ≥ 6.

Proof:

Let C = $c_1, c_2, \ldots, c_n$, where $n \geq 6$ be a geodesic cycle of length greater than equal to 6 in a 3.A.W.C.D critical graph G. Now join c_1 and c_3. Then G+c_1c_3 has a 2-A.W.C.D set Clearly $\{c_1, c_2\}$ cannot dominate G+c_1c_3, since they cannot dominate the vertex c_5. If suppose $\{c_1, x\}$ dominates G+c_1c_3. Then the vertex x must dominates c_3, c_4, c_5 in G+c_1c_3. This implies that x is adjacent to c_1, c_3, c_4, c_5 in G. This implies that $d(c_1, c_4) = 2$ in G. This implies that C is not a geodesic cycle of length greater than or equal to 6. Hence the proof.

Theorem 2.9:

Let G be a 3-A.W.C.D critical graph and v be cut vertex in G. Then there cannot be any dominating set D with $e(v)=1$ in <D>.

Proof:

Let D= $\{u, v, w\}$ be the A.W.C.D set of G, where $u, w \in N_1(v)$. Let x be a pendant vertex adjacent to v. Now join u and w. Then either $\{u, v\}$ or $\{v, w\}$ can only dominate G+uw. Without loss of generality assume that $\{u, v\}$ form a dominating set. But $\{u, v\}$ cannot dominate any of the vertices which are uniquely dominated by w. Hence $\{u, v\}$ cannot be a dominating set. Similarly, $\{v, w\}$ also cannot be a dominating set. Therefore, $\gamma_{ac}(G + uw) = \gamma_{ac}(G)$ only. This is a contradiction to G is 3-A.W.C.D critical. Thus, both u and w cannot be adjacent to v in <D>.

Proposition 2.1:

If G is a k- A.W.C.D critical graph, then no two pendant vertices of G have a common neighbour.

Corollary 2.2:

Any k- A.W.C.D critical graph has at most k-pendant vertices.

Theorem 2.10:

Any 3-A.W.C.D critical graph has at most two cut vertices.

Proof:

If it has three cut vertices, then the diameter of the graph is at least 4, which is a contradiction to the fact that the diameter of any 3-A.W.C.D set is at most 3.

Theorem 2.11:

Any 3-A.W.C.D critical graph has at most one cut vertex.

Theorem 2.12:

If G is a connected 3-A.W.C.D critical graph with a cut vertex, then V(G)= V(A) $\cup$ V(B) such that A= $\{v\}$ and B is a self-centred graph of diameter 2 and the vertex v of A is adjacent to exactly one vertex of B.

Theorem 2.13:

If S is an independent set of r vertices in the connected 3-A.W.C.D critical graph G, then all the vertices of S has degree greater than or equal to r – 1.

Theorem 2.14:

Diameter of a k-A.W.C.D critical graph is at most k.

Theorem 2.15:

Any k-A.W.C.D critical graph can have at most (k-1) number of pendant vertices.

Proof:

Let $p_1,p_2,\ldots p_n$ be the pendant vertices of a k-A.W.C.D critical graph G. Let $q_1,q_2,\ldots.q_k$ be the supports of $p_1,p_2,\ldots\ldots p_k$ respectively. Then clearly $\{q_1,q_2,\ldots q_k\}$ is the only A.W.C.D set of G. Consider any two adjacent vertices of this A.W.C.D set. Without loss of generality assume that p_1 and p_2 are adjacent vertices of G. Now join p_1 and p_2. Then clearly any vertex of A.W.C.D set of G+p_1p_2 contains either p_1 or p_2. We can assume an A.W.C.D set A which contains p_1. Surely A contains $q_3,q_4,\ldots q_k$ also, since still they are cut vertices of G+p_1p_2. As A is connected, the connection between p_1 and the other vertices $q_3,q_4,\ldots q_k$ can be made through q_1 only. Therefore, A must contains $p_1,q_1,q_3,\ldots,q_k$. This implies $|A|\geq$ k, which is a contradiction to the fact that G is k-A.W.C.D critical. Hence G cannot have more than (k-1) number of pendant vertices.

Theorem 2.16:

There exists no A.W.C.D critical tree.

Proof:

Let T be a tree and let u and v be the pendant vertices of T. Let u' and v' be the support of u and v respectively. Now in T+uv any dominating set contains either u, u' and the internal vertices other than v in T or v, v' and the internal vertices other than u in T or all the internal vertices of T. In all the cases γ_{ac} (T+uv) = γ_{ac}(T) only. Which is a contradiction to T is critical. Hence the proof.

Now we define acyclic weak convex domatic Number of Graphs and study the properties of graphs related to it.

Definition 3.2 :

A partition $\{V_1, V_2, \ldots V_n\}$ of the vertex set V is said to be an acyclic weak convex restrained domination partition if each of the V_i is an A.W.C.R.D set.

The cardinality of the minimum A.W.C.R.D partition of V(G) is said to be acyclic weak convex restrained domatic number of G and is denoted by $d_{ac}(G)$.

As acyclic W.C.D set is also a W.C.D set, the following theorems follows trivially:

Theorem 2.17:

For any graph G, $d_{ac}(G) \leq \delta + 1$.

Theorem 2.18:

$d_{ac}(G) = \delta + 1 \Leftrightarrow G = Kp$

Corollary 2.3:

For any graph $G \neq K_p$, $d_{ac}(G) \leq \delta$.

Proposition 2.2:

For any graph, $\gamma_{ac} \times d_{ac} \leq p$.

Theorem 2.19:

Let G be a graph and D be an A.W.C.D set. Then the radius of the induced graph <D> induced by D is at the lowest by r-1, where r is the radius of G.

Theorem 2.20:

For any graph on p vertices G with radius r, $d_{ac}(G) \leq [p/(2r-2)]$.

Theorem 2.21:

For any graph G on p vertices with diameter d, $d_{ac}(G) \leq [p/(d-1)]$.

Question:

For a given two integers d, n with d<n, is it possible to construct a graph with diameter d and A.W.C.D partition n?

Consider n paths of length d-1:

$P_1(p_{1,1},p_{1,2},\ldots, p_{1,d})$, $P_2(p_{2,1},p_{2,2},\ldots, p_{2,d})$, …, $P_n(p_{n,1},p_{n,2},\ldots, p_{n,d})$

For any fixed j $(1 \leq j \leq d)$ join $P_{i,j}$ to $P_{k,j}$ $(1 \leq i, k \leq n)$ and $i \neq k$.

Then clearly each path will form an A.W.C.D set. Also the above graph has diameter d. Hence the above graph has nd number of vertices.

Proposition 2.3:

If a graph G on p vertices contains m number vertices of degree n-1 then $d_{ac} \leq m+\lfloor p/2 \rfloor$.

Proposition 2.4:

If both G and $\overline{G}$ are self-centred of diameter 2, then $d_{ac} \leq \lfloor p/3 \rfloor$.

Proposition 2.5:

For any graph G on p vertices, $d_{ac}(G) + d_{ac}(\overline{G}) \leq p+1$.

Corollary 2.4:

Let G be a graph on p vertices, then $d_{ac}(G) + d_{ac}(\overline{G})=p+1 \Leftrightarrow G = K_p$ or $\overline{K_p}$.

Corollary 2.5:

For any graph $G \neq K_p$ or $\overline{K_p}$, $d_{ac}(G) + d_{ac}(\overline{G}) \leq p - 1$.

Proposition 2.6:

For any cycle C_n, where $n \geq 5$, $d_{ac}(C_n)+d_{ac}(\overline{C_n}) \leq \lfloor n/2 \rfloor +1$.

Theorem 2.23:

For any graph $\gamma_{ac} + d_{ac} \leq n + 1$.

Theorem 2.24:

For any graph $G \neq K_p$ or $\overline{K_p}$, $d_{ac}(G) \times d_{ac}(\overline{G}) \leq (p-1)^2/4$.

Theorem 2.25:

If $G \neq K_p$ is regular, then $d_{ac} = \delta \Leftrightarrow$ there exists a δ - edge A.W.C dominating partition for G.

Corollary 2.6:

If G is k-regular with $d_{ac}(G) = k$, then the number of edges $q = k(k-1)$.

Proof:

Let G be a k-regular graph with $d_{ac}(G) = k$.

From the above theorem, we have $|V(G)| = 2k$. Thus G is a graph on 2k vertices with degree k for all the vertices. This implies that $q=\frac{k(k-1)}{2}+\frac{k(k-1)}{2}=k(k-1)$.

Corollary 2.7:

If G is k-regular and $d_{ac}(G) = k$, then G is a self-centred graph of diameter 2.

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Dom-Chromatic Sets in Bipartite Graphs

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Abstract: *Let G be a simple graph with vertex set V and edge set E. A subset S of V is said to be a dom-chromatic set (or dc-set) if S is a dominating set and the chromatic number of the graph induced by S is the chromatic number of G. The minimum cardinality of a dom-chromatic set in a graph G is called the dom-chromatic number (or dc- number) and is denoted by $\gamma_{ch}(G)$. In this paper, bounds for dom-chromatic numbers for bipartite graphs are discussed.*

Keywords: *Dominating set, domination number, dom-chromatic set (or dc-set), dom-chromatic number (or dc-number).*

1. Introduction

In this paper, we discuss finite, simple undirected graphs. For any graph G, V denotes the vertex set, E denotes the edge set and p denotes the number of vertices. For any graph theoretic terminology which is not defined refer to Harary [2].

The chromatic number $\chi(G)$ is the minimum k such that vertices of G is properly k-colorable. A graph G is said to be a vertex-color-critical graph if $\chi(G - u) < \chi(G)$ for every $u \in V$ and edge-critical if $\chi(G - e) < \chi(G)$ for every $e \in E$. Clearly, edge-critical graphs are vertex-color-critical. In general, any element t of the set $V(G) \cup E(G)$ is critical if $\chi(G - t) < \chi(G)$. A graph is called a color-critical graph if each of its vertices and edges are critical. It is to be noted that the only k-critical graphs for k = 1, 2 and 3 are K_1, K_2 and odd cycles, respectively.

A set $S \subseteq V$ is a dominating set of G if for each $u \in V - S$, there exists a vertex $v \in S$ such that u is adjacent to v. The minimum cardinality of a dominating set in G is called the domination number of G, denoted by $\gamma(G)$. Harary and Haynes [3] defined the conditional domination number $\gamma(G: P)$ as the smallest cardinality of a dominating set S $\subseteq$ V such that the sub graph <S> induced by S satisfies a graph property P. A dominating set $S \subseteq V$ of G is a global dominating set if S is also a dominating set in the complement $\overline{G}$ of G.

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In this paper, we introduce a new conditional dominating set called dom-chromatic set or simply a dc-set which combines domination and coloring property of a graph. A subset S of V is said to be a dom-chromatic set (or dc-set) if S is a dominating set and $\chi(<S>) = \chi(G)$. The minimum cardinality of a dom-chromatic set in a graph G is called the dom-chromatic number (or dc- number) and is denoted by $\gamma_{ch}(G)$.

2. Preliminary results

This section contains some results about domination and some preliminary results on dom-chromatic number which will be used in the next section to prove the main result.

Theorem 2.1 [5, pp 41]: If a graph G has no isolated vertices, then $\gamma(G) \leq \left\lfloor \frac{p}{2} \right\rfloor$.

Theorem 2.2: [5, pp 50]: For any graph G, $\left\lfloor \frac{p}{\Delta(G)+1} \right\rfloor \leq \gamma(G) \leq p - \Delta(G)$.

In a dom-chromatic set, the following observations are made.

Observation 2.3:

(i) Dom-chromatic set exists for all graphs.

(ii) Vertex set V is a trivial dom-chromatic set.

(iii) If S is a dom-chromatic set of G, then each vertex of V – S is not adjacent to at least one vertex of S.

(iv) A dom-chromatic set of a graph is a global dominating set.

Proof:

(i) and (ii) follow trivially.

(iii) Suppose S is a dom-chromatic such that $x \in V - S$ is adjacent to each vertex of S, then $\chi(G) \geq \chi(<S>) + 1 = \chi(G) + 1$ which is a contradiction.

(iv) Let S be a dom-chromatic of a graph G. From (iii), in $\overline{G}$ each vertex of V – S is adjacent to at least one vertex of S. Hence, S is a dominating set of $\overline{G}$ and the result follows.

Proposition 2.4: A dom-chromatic set S is minimal if and only if for each $u \in S$, at least one of the following conditions hold.

(i) $\chi(<S - u>) < \chi(G)$.

(ii) S – u is not a dominating set.

The dom-chromatic number for some standard graphs:

Proposition 2.5:

(i) $\gamma_{ch}(K_n) = n$

(ii) $\gamma_{ch}(nK_1) = n$;

(iii) $\gamma_{ch}(K_{m,n}) = 2$

(iv) $$\gamma_{ch}(P_n) = \begin{cases} (n+3)/3, & \text{if } n \equiv 0 \pmod 3 \\ (n+2)/3, & \text{if } n \equiv 1 \pmod 3 \\ (n+4)/3, & \text{if } n \equiv 2 \pmod 3 \end{cases}$$

(v) a. If n is odd, then $\gamma_{ch}(C_n) = n$.

b. If n is even, then

$$\gamma_{ch}(C_n) = \begin{cases} (n+3)/3, & \text{if } n \equiv 0 \pmod 3 \\ (n+2)/3, & \text{if } n \equiv 1 \pmod 3 \\ (n+4)/3, & \text{if } n \equiv 2 \pmod 3 \end{cases}$$

(vi) $$\gamma_{ch}(W_n) = \begin{cases} 3, & \text{if n is odd} \\ n, & \text{if n is even.} \end{cases}$$

Proposition 2.6: If G is a disconnected graph with k components $G_1, G_2 \ldots, G_k$, then $\gamma_{ch}(G) = \gamma_{ch}(G_m) + \sum_{\substack{i=1 \\ i \neq m}}^{k} \gamma(G_i)$, where $\gamma_{ch}(G_m) = \min_{1 \leq i \leq k} \{\gamma_{ch}(G_i) : \chi(G_i) = \chi(G)\}$, for $m \in \{1, 2\ldots, k\}$.

Proposition 2.7: If G is any connected graph, then $\gamma_{ch}(G) = p - q$ if and only if $G = K_1$.

Proof: Necessary condition is trivial. Suppose that $\gamma_{ch}(G) = p - q$. Since $\gamma_{ch}(G) \geq 1$, $p - q \geq 1$. Further, as G is connected, $p - q \leq 1$. Thus, $p - q = 1$ and hence $G = K_1$.

Proposition 2.8: Let D be any dom-chromatic set of G. Then $|V - D| \leq \sum_{u \in D} deg(u)$.

Proposition 2.9: Let D be any dom-chromatic set of G. Then $|V - D| = \sum_{u \in D} deg(u)$ if and only if $G = pK_1$, $p \geq 1$.

Proof: If $G = pK_1$, then $D = V$ and $\deg(u) = 0$ for each $u \in D$. Then the equality holds. Now suppose that $|V - D| = \sum_{u \in D} \deg(u) = k$.

<u>**Claim:**</u> $k = 0$.

Suppose $k \geq 1$, then two cases arise.

Case i: G is connected.

Then $\chi(G) \geq 2$. Let V – D = {u_1, u2 …, u_k}. Since D is a dominating set, each u_i is adjacent to a vertex of D and hence, contributes at least one degree to D. Since $\chi(<D>) \geq 2$, D contains at least one edge which contributes 2 degrees to D. Hence, $\sum_{u \in D} deg(u) \geq k + 2$, a contradiction.

Case ii: G is disconnected.

If G is totally disconnected, then V = D and hence, |V – D| = k = 0, a contradiction. Hence, G has a non trivial component and <D> contains at least one edge. Then by a similar argument as in case (i), a contradiction arises. Thus in both cases, we arrive at a contradiction, proving that k = 0, i.e., $|V - D| = \sum_{u \in D} deg(u) = 0$. Therefore, V = D and hence, for each u ∈ V, deg(u) = 0. Thus, G is a totally disconnected graph and hence, G = kK_1.

Corollary 2.10: For any non trivial connected graph with a dom-chromatic set D, $\sum_{u \in D} deg(u) \geq |V - D| + 2$.

Proof: If G is vertex-color-critical, then V = D and $\sum_{u \in D} deg(u) = 2q \geq 2 = |V - D| + 2$.

Suppose G is not vertex-color-critical, then since G is non trivial, $\chi(G) \geq 2$. Thus, by a similar argument as in Case (i) of proposition 2.9, $\sum_{u \in D} deg(u) \geq |V - D| + 2$.

Proposition 2.11: For any graph G, $\left\lfloor \frac{p}{\Delta(G)+1} \right\rfloor \leq \gamma_{ch}(G)$ and equality holds if and only if G = pK_1, $p \geq 1$.

Proof: From Theorem 2.2, lower bound is trivial. If G = pK_1, then the result follows. Suppose $\left\lfloor \frac{p}{\Delta(G)+1} \right\rfloor = \gamma_{ch}(G) = k$ and D is a γ_{ch}-set of G.

Case i: G is connected. If $k \geq 2$, then G is a non trivial connected graph. Then by corollary 2.10, $|V - D| < \sum_{u \in D} \deg(u)$. Thus, $p-k < \sum_{u \in D} \deg(u) \leq k\Delta(G)$ and hence, $\frac{p}{\Delta(G)+1} < k$. Hence, $k > \frac{p}{\Delta(G)+1} \geq \left\lfloor \frac{p}{\Delta(G)+1} \right\rfloor = k$, a contradiction. Thus, k = 1 and hence, $\gamma_{ch}(G)=1$. Therefore, G = K_1.

<u>**Case ii:**</u> G is disconnected.

Suppose G is not totally disconnected, then G has atleast one non trivial component. By a similar argument as in case (i), contradiction arises. Therefore, G = pK_1.

Proposition 2.12:

(i)If G is a connected graph, then $\gamma_{ch}(G) = p$ if and only if G is a vertex-color- critical or G is color-critical graph.

(ii)If G is a disconnected graph, then $\gamma_{ch}(G) = p$ if and only if either G is a null graph or has exactly one non trivial component, which is vertex-color-critical or color-critical.

3. Main result

In this section, bipartite graphs are studied. Since the dc-number of a bipartite graph is either $\gamma(G)$ or $\gamma(G) + 1$, graphs with exact bounds are identified. Further certain classes of graphs whose dc-number is half of their order are found with a given diameter.

Proposition 3.1: Let G be a forest with each component of diameter at most 4. Then

(i) If G has at least one component of diameter 3, then $\gamma_{ch}(G) = \gamma(G)$.

(ii) If G has at least one component of diameter 4 with its center adjacent to a pendant vertex, then $\gamma_{ch}(G) = \gamma(G)$.

(iii) If none of the components satisfy (i) or (ii), then $\gamma_{ch}(G) = \gamma(G) + 1$.

Theorem 3.2: If G is a bipartite graph with no isolated vertices, then $\gamma_{ch}(G) \leq \frac{p}{2} + 1$ and $\gamma_{ch}(G) = \frac{p}{2} + 1$ if and only if $G = \frac{p}{2} K_2$.

Proof: Since the dc-number of a bipartite graph is either $\gamma(G)$ or $\gamma(G) + 1$, the upper bound follows. Also, if G is the union of independent edges, then equality holds. Conversely, suppose that $\gamma_{ch}(G) = \frac{p}{2} + 1$ and $\{V_1,V_2\}$ be the vertex partition of V.

<u>**Claim 1:**</u> $|V_1|= |V_2|$

Without loss of generality, let $|V_1| < |V_2|$. Then, $|V_1|< \frac{p}{2}$. Thus, $\gamma_{ch}(G) \leq |V_1| + 1 < \frac{p}{2} + 1$, a contradiction. Hence $|V_1|= |V_2|$.

Let $S = V_1 \cup \{x\}$, $x \in V_2$. Then, S is a dom-chromatic set of G and $|S| = \frac{p}{2} +1$. Hence, S is a γ_{ch}-set of G.

Claim 2: $|E(<S>)| = 1$

Suppose $|E(<S>)| \geq 2$, then x is adjacent to more than one vertex of V_1. Let $N_{<S>}(x) = \{x_1, x_2,\ldots,x_r\}$, $r \geq 2$. Then for each i, $1 \leq i \leq r$, $S - x_i$ induces a 2-chromatic graph. Since S is minimal, $S - x_i$ cannot be a dominating set of G. Thus, there exists a unique vertex $y_i \in V_2$ such that $x_iy_i \in E$. Similarly, as S is maximal, for each $z \in V_1 - N[x]$ there exists a unique vertex $z' \in V_2$ such that $zz' \in E$. Then, $|V_2| \geq \frac{p}{2} + 1$, a contradiction. Thus, each $x \in V_2$, x is adjacent to only one vertex of V_1. By claim 1, $\deg(x) = 1$ for each $x \in V_1 \cup V_2$. Therefore, $G = mK_2$.

Corollary 3.3: If G is a bipartite graph with no isolated vertices and $\gamma_{ch}(G) = \frac{p}{2} + 1$, then $\gamma(G) = \frac{p}{2}$.

Proposition 3.4:

(i) A bipartite graph G has a dominating edge if and only if $\gamma_{ch}(G) = 2$

(ii) If G is a tree of diameter 2 or 3 then, $\gamma_{ch}(G) = 2$.

Proof:

(i) Let $e = xy$ be a dominating edge of G. Then $\{x, y\}$ is a γ_{ch}-set of G. Conversely suppose $\gamma_{ch}(G) = 2$ and S be any γ_{ch}-set of G, then $|S| = 2$. Since $\chi(<S>) = 2$, $<S> = K_2$. Further, S is also a dominating set of G implies G has a dominating edge.

(ii) Since $\text{diam}(G) = 2$ or 3, G has a dominating edge. Then by (i), $\gamma_{ch}(G) = 2$.

Theorem 3.5: If T is a tree with $\text{diam}(G) \leq 4$, then $\gamma_{ch}(G) = \frac{p}{2}$ if and only if G is $K_{1,3}$, P_4 or a graph in the family given in figure 1.

Proof: Necessary condition is trivial. Suppose that $\gamma_{ch}(G) = \frac{p}{2}$. Clearly, $\text{diam}(G) \geq 2$. For otherwise, $G = K_2$ and then $\gamma_{ch}(G) = p$, a contradiction.

Case i: $\text{diam}(G) = 2$.

Then $G = K_{1,n}$, $n \geq 2$, which implies $\gamma_{ch}(G) = 2$. Then $p = 4$. Therefore, $G = K_{1,3}$.

Case ii: $\text{diam}(G) = 3$.

From Proposition 3.5(ii), $\gamma_{ch}(G) = 2$. Therefore, $p = 4$ and hence, $G = P_4$.

Case iii: $\text{diam}(G) = 4$.

Then there exists a unique vertex x such that $e(x) = 2$. Therefore, N(x) can be partitioned into three sets as follows:

$S_1 = \{y \in N(x) \mid deg(y) = 1\}$

$S_2 = \{y \in N(x) \mid deg(y) = 2\}$

$S_3 =\{y \in N(x) \mid deg(y) \geq 3\}$

Let $|S_i| = m_i$, $1\leq i \leq 3$.

Claim 1: $m_1 > 0$, $m_2 > 0$ and $m_3 > 0$ cannot be simultaneously hold

Suppose not. Then $\gamma_{ch}(G) = m_2 + m_3 +1$ and $p \geq 1 + m_1 + 2m_2 + 3m_3$, a contradiction to $\gamma_{ch}(G) = \frac{p}{2}$.

As diam(G) = 4, at least one of m_2 or m_3 is not 0.

Claim 2: $m_1 > 0$, $m_2 = 0$ and $m_3 > 0$ cannot simultaneously hold.

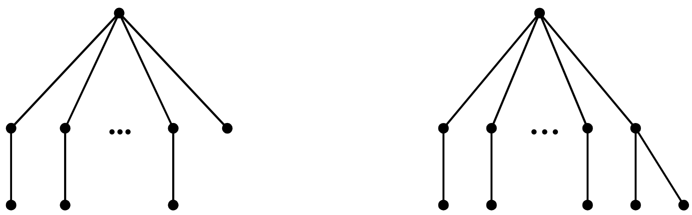
Suppose claim does not hold. Then $\gamma_{ch}(G)= m_3 + 1$ and $p \geq 3m_3 + m_1 + 1$, a contradiction.

Sub Case i: $m_1 > 0$

By claim 1, one of m_2 and m_3 is zero. By claim 2, m_2 cannot be zero. Hence, $m_2 > 0$ and $m_3 = 0$. Since diam(G) = 4, $m_2 \geq 2$ and $\gamma_{ch}(T) = m_2 + 1$. Hence, $p = 2m_2 + 2= m_1+2m_2+1$. Therefore, $m_1 = 1$. Then G is in the family given in Figure 1(a).

Sub Case ii: $m_1 = 0$.

Suppose $m_2 = 0$. Then $m_3 \geq 2$ and $\gamma_{ch}(G) = m_3 + 1$. Therefore, $p = 2m_3 + 2$. Also $p \geq 3m_3 + 1$ implies $1 \geq m_3$, a contradiction. So let $m_3 = 0$. Then, $m_2 \geq 2$ and $p = 2m_2 + 1$. This implies G is a tree with odd number of vertices, a contradiction. Hence, both m_2 and m_3 are not zero. Then $\gamma_{ch}(G) = m_2 + m_3 + 1$ and hence, $p = 2m_2 + 2m_3 + 2$. Also $p \geq 2m_2 + 3m_3 + 1$ implies $1 \geq m_3$. Therefore, $m_3 = 1$ and hence, $\gamma_{ch}(G) = m_2 + 2$. Let $y \in S_3$. But, $p = 2m_2 + 4$ implies deg(y) = 3. Then G is a tree in the family given in Figure 1(b).



(a) (b)

Figure 1

Theorem 3.6: Suppose G is a forest with each component of diameter 1 or 2. Then $\gamma_{ch}(G)= \frac{p}{2}$ if and only if G is either $K_{1,3} \cup mK_2$, $m > 0$ or $2P_3 \cup mK_2$, $m \geq 0$.

Proof: Necessary condition can be easily verified. Conversely, suppose $\gamma_{ch}(G)= \frac{p}{2}$. Let $G_1, G_2 ..., G_r$, $r \geq 2$ be the components of G.

Claim 1: All components cannot be of diameter 1.

Suppose not. Then $G = mK_2$, $m \geq 1$. By Theorem 3.2, $\gamma_{ch}(G) = m + 1 = \frac{p}{2} + 1$, a contradiction.

<u>**Case i:**</u> All components are of diameter 2.

Then for each i, $1 \leq i \leq r$,

$$G_i = K_{1,n_i}\ \ n_i \geq 2$$

$\Rightarrow$ $\gamma_{ch}(G) = r + 1$ ----------------- (1)

$\Rightarrow$ $p = 2r + 2$ ----------------- (2)

<u>**Claim 2:**</u> G cannot have K_{1,n_i}, $n_i \geq 4$ as a sub graph.

Suppose $G_1 = K_{1,n_1}$, $n_1 \geq 4$. From (1), $|V(G_i)| \geq 3$ for each i. Then, $p \geq 5 + 3(r - 1) = 3r + 2$, a contradiction to (2).

<u>**Claim 3:**</u> G cannot have more than 2 components.

Suppose $r \geq 3$. Then $p \geq 3r \geq 2r + 3$, contradiction to (2).

As $r = 2$, from (2), $p = 6$. Since both components are of diameter 2, $G = 2P_3$ and (ii) holds.

<u>**Case ii :**</u> G has components of diameter 1 and diameter 2.

Hence, each components is either a $K_{1,n}$, $n \geq 2$ or a K_2. Let r_1, r_2 be the number of components of $K_{1,n}$ and K_2 respectively. Then $\gamma_{ch}(G) = r_1 + r_2 + 1$. Therefore,

$p = 2r_1 + 2r_2 + 2$ ----------------- (3)

<u>**Claim 4:**</u> $r_1 \leq 2$.

Suppose not. Then $p \geq 3r_1 + 2r_2 > 2r_1 + 2r_2 + 2$, a contradiction to (3).

Suppose $r_1 = 1$. Then $\gamma_{ch}(G) = r_2 + 2$ and $p = 2r_2 + 4$. Hence, $K_{1,n}$ can have only 4 vertices. Then $G = K_{1,3} \cup mK_2$, $m > 0$ and (i) holds. If $r_1 = 2$, then $\gamma_{ch}(G) = r_2 + 3$. Then $p = 2r_2 + 6$. Since two $K_{1,n}$'s with $n \geq 2$ and 6 vertices is $2P_3$, $G = 2P_3 \cup mK_2$ and (iii) holds.

Theorem 3.7. If G is a forest of even order without isolated vertices and each component is of diameter at most 3 with at least one component of diameter 3, then $\gamma_{ch}(G) = \frac{p}{2}$ if and only if G is

i. mP_4, $m > 1$ or

ii. $mP_4 \cup nK_2$, $m, n \geq 1$.

Proof: Necessary condition is easily verified. Suppose $\gamma_{ch}(G) = \frac{p}{2}$. Let r_1, r_2 and r_3 be number of components of diameter 1, 2 and 3 respectively. Then

$p \geq 2r_1 + 3r_2 + 4r_3$ ----------------- (1)

and $r_3 > 0$ ----------------- (2)

<u>**Claim 1**</u>: $r_1 = 0$ and $r_2 > 0$ cannot hold.

Suppose not. Then $\gamma_{ch}(G) = r_2 + 2r_3$ and hence, $p = 2r_2 + 4r_3$. From (1), $p \geq 4r_3 + 3r_2$, a contradiction.

<u>Claim 2</u>: $r_1 > 0$ and $r_2 > 0$ cannot hold.

Suppose not. Then $\gamma_{ch}(G) = r_1 + r_2 + 2r_3$, which implies $p = 2r_1 + 2r_2 + 4r_3$ a contradiction to (1).

From claim 1, claim 2 and from (2), only 2 cases are to be considered.

<u>Case i:</u> $r_1 = 0$ and $r_2 = 0$.

Therefore, $\gamma_{ch}(G) = 2r_3$, and hence, $p = 4r_3$. Since each component has at least 4 vertices, $G = r_3P_4$ and (i) is proved.

<u>Case ii</u> : $r_1 > 0$, $r_2 = 0$.

Then $\gamma_{ch}(G) = r_1 + 2r_3$, which implies $p = 2r_1 + 4r_3$ and hence, $G = r_3P_4 \cup r_1 K_2$ and (ii) holds.

Theorem 3.8. If G is a forest with isolated vertices and each non-trivial component is of diameter at most 2 with at least one component of diameter 2, then $\gamma_{ch}(G) = \frac{p}{2}$ if and only if $G = \bigcup_{i=1}^{i=k} K_{1,n_i} \cup rK_2 \cup (\sum_{i=1}^{k} n_i - k - 2)K_1$, $n_i \geq 2$ for each i, $k \geq 1$, $r \geq 0$, $\sum_{i=1}^{k} n_i - k - 2 > 0$.

Proof : Necessary condition is trivial. So suppose $\gamma_{ch}(G) = \frac{p}{2}$.

<u>Case i</u> : All non trivial components are of diameter 2.

Then, $G = \bigcup_{i=1}^{i=k} K_{1,n_i} \cup mK_1$, $n_i \geq 2$, which implies $\gamma_{ch}(G) = k + m + 1$. Therefore, $p = 2k + 2m + 2$. By the structure of G, $p = \sum_{i=1}^{k} n_i + k + m$. Therefore, $m = \sum_{i=1}^{k} n_i - k - 2$ and satisfies the given condition.

<u>Case ii</u>: G contains non trivial components of diameter 1 and 2.

Then $G = \bigcup_{i=1}^{i=k} K_{1,n_i} \cup rK_2 \cup mK_1$, $r > 0$, $m > 0$. Therefore, $\gamma_{ch}(G) = k + r + m + 1$ and hence, $p = 2k + 2r + 2m + 2$. From the structure of G, $p = \sum_{i=1}^{k} n_i + k + 2r + m$.

Therefore, m = $\sum_{i=1}^{k} n_i$ - k - 2 and the result follows.

Notation 3.9. P_4^i is the family of graphs obtained from P_4 by randomly joining i vertices to the intermediate vertices. It can be seen that any tree of diameter 3 is P_4^i, i ≥ 0.

Theorem 3.10. If G is a forest with isolated vertices and the non-trivial components are of diameter at most 3 with at least one component of diameter 3, then $\gamma_{ch}(G) = \frac{p}{2}$ if and only if G is one of the following graphs.

i) $\bigcup_{i=1}^{i=m} P_4^{l_i} \cup rK_2 \cup nK_1$, r ≥ 0, $\sum_{i=1}^{m} l_i = n$

ii) $\bigcup_{i=1}^{i=m} P_4^{l_i} \cup (\bigcup_{j=1}^{s} K_{1,n_j}) \cup rK_2 \cup nK_1$, r ≥ 0, $n_j \geq 2$ for each j and $\sum_{i=1}^{m} l_i + \sum_{j=1}^{s} n_j = s + n$.

Proof: Necessary condition is trivial. Suppose $\gamma_{ch}(G) = \frac{p}{2}$.

<u>**Case i:**</u> All non trivial components are of diameter 3.

Then G = $\bigcup_{i=1}^{i=m} P_4^{l_i} \cup nK_1$. Therefore, p = 4m + $\sum_{i=1}^{m} l_i$ + n. As $\gamma_{ch}(G)$ = 2m + n, p = 4m + 2n. Then $\sum_{i=1}^{m} l_i$ = n. Hence, G satisfies (i) with r = 0.

<u>**Case ii:**</u> Non trivial components are of diameter 3 and 1.

Then G = $\bigcup_{i=1}^{i=m} P_4^{l_i} \cup rK_2 \cup nK_1$. Therefore, p = 4m + $\sum_{i=1}^{m} l_i$ + 2r + n. As $\gamma_{ch}(G)$ = 2m + r + n, p = 4m + 2r + 2n. Then $\sum_{i=1}^{m} l_i$ = n and hence, G satisfies (i) with r > 0.

<u>**Case iii:**</u> Non trivial components are of diameter 3 and 2.

Then G = $\bigcup_{i=1}^{i=m} P_4^{l_i} \cup (\bigcup_{j=1}^{s} K_{1,n_j}) \cup nK_1$, $n_j \geq 2$. This implies that p = 4m + $\sum_{i=1}^{m} l_i$ + $\sum_{j=1}^{s} (n_j + 1)$ + n. But $\gamma_{ch}(G)$ = 2m + s + n implies p = 4m + 2s + 2n. Therefore, $\sum_{i=1}^{m} l_i$ + $\sum_{j=1}^{s} n_j$ = s + n. Then G satisfies (ii) with r = 0.

Case iv: Non trivial components are of diameter 3, 2 and 1.

Then, G = $\bigcup_{i=1}^{i=m} P_4^{l_i} \cup (\bigcup_{j=1}^{s} K_{1,n_j}) \cup rK_2 \cup nK_1$, $n_j \geq 2$ and hence, p = 4m + $\sum_{i=1}^{m} l_i$ + $\sum_{j=1}^{s} (n_j + 1)$ + 2r+ n. Also, $\gamma_{ch}(G)$ = 2m + s + r + n implies p = 4m + 2s + 2r + 2n. This leads to $\sum_{i=1}^{m} l_i$ + $\sum_{j=1}^{s} n_j$ = s + n. Hence, G satisfies (ii) with r > 0.

Theorem 3.11. If G is bipartite, then $\gamma_{ch}(G)$ = p if and only if either G = K_2 or G = $K_2 \cup (p - 2)K_1$.

Proof : If G is connected, then $\gamma_{ch}(G)$ = p if and only if G is vertex-color-critical. Since the only bipartite vertex-color-critical graph is K_2, the result follows. Suppose G is disconnected. The result follows from Proposition 4.1.7(ii).

Theorem 3.12. Let G be a tree of diameter 5. If at least one of the central elements of G is adjacent to a pendant vertex, then $\gamma_{ch}(G) = \gamma(G)$, otherwise $\gamma_{ch}(G) = \gamma(G) + 1$.

Proof: As diam(G) = 5, G has two central elements and they are adjacent. Let them be x and y. Then they are adjacent. Let $S_1 = \{u \mid u \in N(x) - y\}$, $S_2 = \{u \mid u \in N(y) - x\}$, $S_3 = \{u \mid u \in S_1$ and $d(u) > 1\}$ and $S_4 = \{u \mid u \in S_2$ and $d(u) > 1\}$.

Case i: Both x and y are not adjacent to any pendant vertices.

Then $S_1 \cup S_2$ is a γ-set of G and $S_1 \cup S_2 \cup \{x\}$ a γ_{ch}-set of G. Therefore, $\gamma_{ch}(G) = \gamma(G) + 1$.

Case ii: x or y is adjacent to a pendant vertex not both.

Suppose x is adjacent to a pendant vertex. Then $S_2 \cup S_3 \cup \{x\}$ is a γ_{ch}-set as well as a γ-set of G. Therefore, $\gamma_{ch}(G) = \gamma(G)$.

Case iii: Both x and y are adjacent to pendant vertices.

Then $S_3 \cup S_4 \cup \{x, y\}$ is a γ_{ch}-set as well as a γ-set of G. Therefore, $\gamma_{ch}(G) = \gamma(G)$.

Theorem 3.13. Let G be a tree of diameter 5. Then $\gamma_{ch}(G) = \frac{p}{2}$ if and only of G is a graph in the family of trees given in figure 2.

Proof: Necessary condition is trivial. Suppose $\gamma_{ch}(G) = \frac{p}{2}$. Let x and y be the central elements of G. Let $S_1 = \{u \mid u \in N(x) - y\}$ and $S_2 = \{u \mid u \in N(y) - x\}$.

__Case i:__ Both x and y are not adjacent to any pendant vertices.

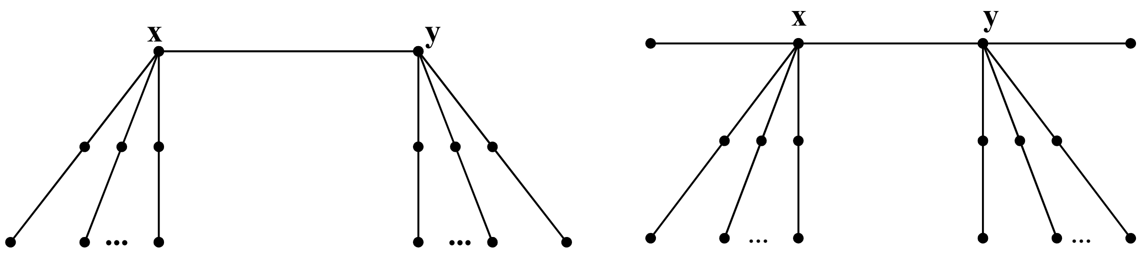
Then $S_1 \cup S_2 \cup \{x\}$ is γ_{ch}-set of G. Then $p = 2(|S_1| + |S_2| + 1)$, which implies that G is a tree in the family given in figure 3(a)

__Case ii:__ At least one of x and y is adjacent to all pendant vertices

First the following claim is proved.

__Claim :__. Only one of x and y is adjacent to pendant vertices cannot hold.

Suppose not. Let x be adjacent to pendant vertices and y is not. Define $S_3 = \{u \mid u \in S_1$ and $d(u) > 1\}$. Then $S_2 \cup S_3 \cup \{x\}$ is a γ_{ch}-set of G. Therefore, $\gamma_{ch}(G) = |S_2| + |S_3| + 1$ and hence, $p = 2(|S_2| + |S_3| + 1)$. Also, from the structure of G, $p \geq 2(|S_2| + |S_3|) + 3$, a contradiction.



(a) **Figure 2** (b)

Hence, by the claim both x and y must be adjacent to pendant vertices. Let $S_4 = \{u \mid u \in S_2$ and $d(u) > 1\}$. Then $S_3 \cup S_4 \cup \{x, y\}$ is a γ_{ch}-set of G. Therefore, $\gamma_{ch}(G) = |S_3| + |S_4| + 2$ and hence, $p = 2(|S_3| + |S_4| + 2)$. Therefore, G is a tree in the family given in figure 2(b).

Proposition 3.14. If G is a bipartite graph of diameter 2, then $\gamma_{ch}(G) = 2$.

Proof : Let xyz be a diameteral path of G. Define 3 sets as follows:

$S_1 = N(x) - y$, $S_2 = N(y) - \{x, z\}$, $S_3 = N(z) - y$. Since G is bipartite, all the above 3 sets induce null graphs. If $S_1 = S_2 = S_3 = \phi$, then $G = K_{1,2}$ and hence, $\gamma_{ch}(G) = 2$. If $S_1 = S_3 = \phi$ and $S_2 \neq \phi$, again G is a $K_{1,n}$, $n \geq 3$. Therefore, $\gamma_{ch}(G) = 2$. So suppose S_1 or $S_3 \neq \phi$, say S_1 .

<u>Claim</u>: $S_1 = S_3$.

Let u $\in S_1 - S_3$. Since diam(G) = 2, d(u, z) = 2. As u is not adjacent to y, there exists a v such that uvz is a path. Then xyzvux is a 5-cycle, a contradiction.

Then {x, y} is a γ_{ch}-set of G.

Proposition 3.15. Let G be a tree of diameter 3. Then $\gamma_{ch}(G) = p - \Delta(G)$ if and only if G = P_4 or G is a tree in the family given in figure 3.

Proof : Clearly, $\gamma_{ch}(G) = 2$. Suppose, G = P_4. Then $\gamma_{ch}(G) = 2$, p = 4, $\Delta(G) = 2$ and the proposition holds. Suppose G is a tree given in Figure 4. Then $\gamma_{ch}(G) = 2$, p = deg(z) + 2 and $\Delta(G)$ = deg(z). Then the result holds.

Conversely, suppose $\gamma_{ch}(G) = p - \Delta(G)$. Let e = yz be a dominating edge. Then deg(y) ≥ 2 and deg(z) ≥ 2.

<u>Claim</u> : deg(y) ≥ 3 and deg(z) ≥ 3 cannot hold simultaneously.

<u>Case i</u>: deg(y) = deg(z) = 2

Then G = P_4 and the conditions are satisfied.

<u>Case ii</u>: deg(y) = 2 and deg(z) ≥ 3.

Let deg(z) = n ≥ 3. Then $\Delta(G)$ = n and p = n + 2. Therefore, $p - \Delta(G) = 2 = \gamma_{ch}(G)$ and G is a tree given in Figure 3.

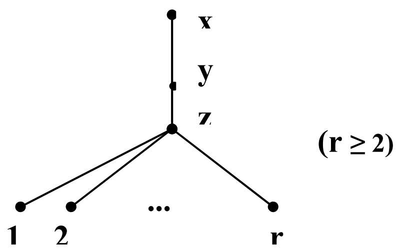


Figure 3

Proposition 3.16. Suppose G is a bipartite unicyclic graph and for each v $\in$ V - V(C), d(v, C) = 1, then $\gamma_{ch}(G) \leq p - t$, where C is the unique cycle and t is the number of pendent vertices of G. Further the equality holds if and only if deg(v) > 2 for each v $\in$ V(C).

Proof : From the given condition, it is clear that any vertex not in C is adjacent to exactly one vertex of C and is a pendant vertex. Then V(C) is a dom-chromatic set of G and |V(C)|= p – t. This gives the given upper bound. Suppose deg(v) >2 for each v $\in$ V(C). Then V(C) is a γ_{ch}-set of G, and the bound is attained. Suppose $\gamma_{ch}(G) = p - t$. If there exists a vertex v in C of degree 2, then V(C) - v is a dom-chromatic set of G of cardinality p - t -1, a contradiction. Thus, deg(v) > 2 for each v $\in$ V(C).

Proposition 3.17. If G is a path, then $\gamma_{ch}(G) = \frac{p}{2}$ if and only if G is P_4, P_6, or P_8.

Proof : Necessary condition is trivial. Conversely, suppose $\gamma_{ch}(G) = \frac{p}{2}$.

<u>**Case i:**</u> $p \equiv 0 \pmod 3$

From Proposition 4.1.5 (iv), $\gamma_{ch}(G) = \frac{p+3}{3}$. Then p = 6.

<u>**Case ii**</u>: $p \equiv 1 \pmod 3$

From Proposition 4.1.5(iv), $\gamma_{ch}(G) = \frac{p+2}{3}$. Then p = 4.

<u>**Case iii:**</u> $p \equiv 2 \pmod 3$

From Proposition 4.1.5(iv), $\gamma_{ch}(G) = \frac{p+4}{3}$. Then p = 8.

Proposition 3.18. If G is an even cycle, then $\gamma_{ch}(G) = \frac{p}{2}$ if and only if G is C_4, C_6, or C_8.

Theorem 3.19. If G is any (p, q) graph, q ≥ 1, then $\gamma_{ch}(G) = p - q + 1$ if and only if G contains exactly $\gamma_{ch}(G)$ -1 components and exactly one of the following holds.

i. each component is isomorphic to $K_{1,s}$'s, s ≥ 0

ii. exactly one component is a tree with diameter 3 or $K_{1,t}$, t ≥ 1 and every other component is isomorphic to $K_{1,m}$'s, m ≥ 0

Proof: Suppose G has $\gamma_{ch}(G)$ - 1 components with (i) or (ii) is satisfied. Let $\gamma_{ch}(G) - 1 = k$ and $G = G_1 \cup G_2 \cup .. \cup G_k$. Let each G_i be a (p_i, q_i)-graph. Then in both cases, $\gamma_{ch}(G) = k + 1$ and $q = \sum_{i=1}^{k} q_i = \sum_{i=1}^{k} (p_i - 1) = p - k$ and hence, $\gamma_{ch}(G) = p - q + 1$.

Conversely, suppose $\gamma_{ch}(G) = p - q + 1$. Suppose G has k components.

<u>**Claim 1:**</u> $k = \gamma_{ch}(G) - 1$.

Since q ≥ 1, $\chi(G) \geq 2$ and a γ_{ch}-set contains at least one vertex from each component,

$$k + 1 \leq \gamma_{ch}(G) \quad \text{---------------------} \quad (1)$$

Since each component is connected, q ≥ p - k, and thus, $k \geq p - q = \gamma_{ch}(G) - 1$. Therefore, $k + 1 \geq \gamma_{ch}(G)$. Therefore, from (1), $k + 1 = \gamma_{ch}(G)$.

Let $G_1, G_2,\ldots,G_k$ be the components of G. Without loss of generality, let $\gamma_{ch}(G_1) = \min_{1 \leq i \leq k} \{\gamma_{ch}(G_i) \mid \chi(G_i) = \chi(G)\}$. From Claim 1, $\gamma_{ch}(G) - 1 = k$.

<u>**Claim 2:**</u> $\gamma_{ch}(G_1) = 2$ and $\gamma_{ch}(G_i) = 1$, i ≥ 2.

Since G contains an edge, $\gamma_{ch}(G_1) \geq 2$. Suppose $\gamma_{ch}(G_1) \geq 3$. By Claim 1, $\sum_{i=2}^{k} \gamma(G_i) \geq k - 1 \geq \gamma_{ch}(G) - 2$ and therefore, $\gamma_{ch}(G) = \gamma_{ch}(G_1) + \sum_{i=2}^{k} \gamma(G_i) \geq \gamma_{ch}(G) + 1$, a contradiction.

Thus, $\gamma_{ch}(G_1) = 2$. Now $\gamma_{ch}(G) = 2 + \sum_{i=2}^{k} \gamma(G_i)$. Therefore, $\sum_{i=2}^{k} \gamma(G_i) = \gamma_{ch}(G) - 2 = k - 1$.

$\gamma(G_i) = 1$, for each i.

<u>Claim3:</u> Each G_i is a tree.

Suppose G_j contains a cycle. Then $q_j \geq p_j$ and $q_i \geq p_i - 1$ for each $i \neq j$. Now, $q = \sum_{i=1}^{k} q_i = q_j + \sum_{\substack{i=1 \\ i \neq j}}^{k} q_i \geq p_j + \sum_{\substack{i=1 \\ i \neq j}}^{k} (p_i - 1) = \sum_{i=1}^{k} p_i - (k - 1) = p - \gamma_{ch}(G) + 2$.

Thus, $\gamma_{ch}(G) \geq p - q + 2$, a contradiction.

Each G_i is a tree and $\gamma(G_i) = 1$ imply that $G_i = K_{1, s}$, $s \geq 0$ for each $i \neq 1$. Since $G_1 = K_{1, s}$, $s \geq 0$ and $\gamma_{ch}(G_1) = 2$ imply that either G_1 is a $K_{1, s}$, $s \geq 1$ or a tree with diameter 3.

Theorem 3.20. If T is a tree with diam(T) = 4 and k is the number of non pendant vertices of T, then $\gamma_{ch}(T) = k$.

Proof : Since diam(T) = 4, T has unique center. Let u be the center of T and $S = \{x \mid x \in N(u), \deg(x) \geq 2\}$. If u is adjacent to a pendant vertex, then $S \cup \{u\}$ is a γ_{ch}-set of T. Hence, $\gamma_{ch}(T)$ is the number of non pendant vertices of T and let it be k. If u is not adjacent to any pendant vertex, then again $S \cup \{u\}$ is a γ_{ch}-set of T and the result follows.

Proposition 3.21. If G is a tree of diameter 3, then $\gamma_{ch}(G) = \gamma_{ch}(\overline{G})$ if and only if $G = P_4$.

Proof : If G is P_4, then the result is trivial. Conversely suppose $\gamma_{ch}(G) = \gamma_{ch}(\overline{G})$. Let the dominating edge of G be e = uv. Let deg(u) = m and deg(v) = n. Also in $\overline{G}$, $(N(u) - v) \cup (N(v) - v)$ is a K_{m+n-2} and $\chi(\overline{G}) = m + n - 2$. Clearly, the above set is dominating $\overline{G}$. Hence, $\gamma_{ch}(\overline{G}) = m + n - 2$. Since $\gamma_{ch}(G) = \gamma_{ch}(\overline{G})$, $m + n = 4$ and the result follows.

Proposition 3.22. If G and $\overline{G}$, both bipartite with diameter 3, then $\gamma_{ch}(G) = \gamma_{ch}(\overline{G})$ if and only if G = P_4.

Solution: If G = P_4, then from Proposition 3.21, the equality holds. Conversely let the equality hold and uvwx be a diameteral path in G. Then N(v) and N(w) induce null graphs. Further, as $\overline{G}$ is bipartite both N(v) and N(w) are of cardinality 2.

<u>**Case i:**</u> both u and x are pendant vertices.

Then G = P_4 and hence, G is P_4.

<u>**ase ii:**</u> at least one of u and x is non pendant vertex.

Suppose u is a non pendant vertex and then by similar argument |N(u)| = 2 and N(u) is a null graph. Let N(u) = {v, u_1}. Clearly, u_1 is not adjacent to x, otherwise a 5-cycle is induced. Then

{ u_1, v, x} induces C_3, a contradiction.

Thus, from case i and ii, a solution is obtained only when G is P_4.

Theorem 3.23. If G is a tree of diameter 3, then $\gamma_{ch}(G) + \gamma_{ch}(\overline{G}) = p$.

Since G is a tree of diameter 3, G has a dominating edge. Therefore, $\gamma_{ch}(G) = 2$. Let uv be the dominating edge of G and V_1, V_2 be the set of pendant vertices adjacent to u and v respectively. Then in $\overline{G}$, $\langle V_1 \cup V_2 \rangle$ induces a complete graph K_{p-2} and, u and v are non adjacent. Further u is adjacent to each vertex of V_1 and v is adjacent to each vertex of V_2 in $\overline{G}$. Thus, $V_1 \cup V_2$ is a γ_{ch}-set of $\overline{G}$. Therefore, $\gamma_{ch}(\overline{G}) = p - 2$ and (i) follows.

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On The Complement of the Boolean Function Graph $B(\overline{K_p}$, NINC, L(G)) of a Graph

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Abstract: *For any graph G, let V(G) and E(G) denote the vertex set and edge set of G respectively. The Boolean function graph B($\overline{K}_p$, NINC, L(G)) of G is a graph with vertex set V(G)∪E(G) and two vertices in B($\overline{K}_p$, NINC, L(G)) are adjacent if and only if they correspond to two adjacent edges of G or to a vertex and an edge not incident to it in G. For brevity, this graph is denoted by $B_2(G)$. In this paper, structural properties of the complement $\overline{B_2}(G)$ of $B_2(G)$ including traversability and eccentricity properties are studied. Also covering, independence and chromatic numbers are determined.*

1. Introduction

Graphs discussed in this paper are undirected and simple graphs. For a graph G, let V(G) and E(G) denote its vertex set and edge set respectively. *Eccentricity* of a vertex $u\in V(G)$ is defined as $e_G(u) = \max \{d_G(u, v): v\in V(G)\}$, where $d_G(u, v)$ is the distance between u and v in G. If there is no confusion, then we simply denote the eccentricity of vertex v in G as e(v) and d(u, v) to denote the distance between two vertices u, v in G respectively. The minimum and maximum eccentricities are the *radius* and *diameter* of G, denoted r(G) and diam(G) respectively. When diam(G) = r(G), G is called a *self-centered* graph with radius r, equivalently G is r-self-centered. A vertex u is said to be an eccentric point of v in a graph G, if d(u, v) = e(v). In general, u is called an *eccentric point*, if it is an eccentric point of some vertex. We also denote the i^{th} neighborhood of v as $N_i(v)=\{u\in V(G) : d_G(u , v) = i\}$. A connected graph G is said to be *geodetic*, if a unique shortest path joins any two of its vertices.

A vertex and an edge are said to *cover* each other, if they are incident. A set of vertices, which covers all the edges of a graph G is called a *point cover* for G. The smallest number of vertices in any point cover for G is called its *point covering number* and is

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denoted by $\alpha_0(G)$ or α_0. A set of vertices in G is *independent*, if no two of them are adjacent. The largest number of vertices in such a set is called the *point independence number* of G and is denoted by $\beta_0(G)$ or β_0.

The *Boolean function graph* B($\overline{K_p}$, NINC, L(G)) G is a graph with vertex set $V(G)\cup E(G)$ and two vertices in B($\overline{K_p}$, NINC, L(G)) are adjacent if and only if they correspond to two adjacent edges of G or to a vertex and an edge not incident to it in G. For brevity, this graph is denoted by $B_2(G)$. In other words, $V(B_2(G)) = V(G)\cup V(L(G))$; and $E(B_2(G)) = [E(T(\overline{G}))-(E(\overline{G})\cup E(\overline{L(G)}))]\cup E(L(G)$, where $\overline{G}$, L(G) and T(G) denote the complement, the line graph and the total graph of G respectively. The vertices of G and L(G) in $B_2(G)$ are referred as point and line vertices respectively and the line vertex in $B_2(G)$ corresponding to an edge e in G is denoted by e'. In this paper, we study structural properties of the complement $\overline{B_2(G)}$ of $B_2(G)$ including traversability and eccentricity properties. The definitions and details not furnished in this paper may found in [1].

The mixed relations of incident, non-incident, adjacent and non-adjacent can be used to analyze nature of clustering of elements of communication networks. The concept of chromatic number could be used in a particular type of clustering network such that each cluster is either independent in that nework. Also any other clustering of network with each cluster having at lest one colour class element of vertices network.

2. Prior Results

Theorem 2.1 [1]: For any nontrivial connected graph G, $\alpha_0 + \beta_0 = p$, where p is the number of vertices in G.

3. Main Results

The following elementary properties of the complement $\overline{B_2(G)}$ of the Boolean function graph $B_2(G)$ of a graph G are immediate. Let G be a (p, q) graph.

Observation:

3.1: Degree of a point vertex v in $\overline{B_2(G)}$ is $p - 1 + \deg_G(v)$ and the degree of line vertex e' is $q + 1 - \deg_{L(G)}(e')$.

3.2: $\overline{B_2(G)}$ is connected, for any graph G.

3.3: $\overline{B_2}(G)$ is biregular if and only if G is regular and is regular if and only if G $\cong$ nK_1, for $n\geq 2$.

3.4: No vertex of $\overline{B_2}(G)$ is a cut-vertex.

3.5: Girth of $\overline{B_2}(G)$ is 3.

3.6: Maximum number of edge disjoint triangles in $\overline{B_2}(G)$ is q and each vertex of $\overline{B_2}(G)$ lies on a triangle.

3.7: Each edge of $\overline{B_2}(G)$ lies on a triangle if and only if each edge of $\overline{L}(G)$ lies on a triangle. In this case, L($\overline{B_2}(G)$) is Hamiltonian.

3.8: If $K_2 \cup K_1$ is a sub graph of G, then $\overline{B_2}(G)$ contains K_4-e as an induced sub graph and hence not geodetic. Therefore, $\overline{B_2}(G)$ is geodetic if and only if G $\cong$ nK_1 or K_2, for $n \geq 2$.

In the following, a characterization of $\overline{B_2}(G)$ to be Eulerian is given.

Theorem 3.1: Let G be any (p, q) graph with q odd. Then $\overline{B_2}(G)$ is Eulerian if and only if degree of each vertex in G is of same parity.

Proof: Assume q is odd and $\overline{B_2}(G)$ is Eulerian. Then degree of each vertex in $\overline{B_2}(G)$ is even.

Case(i): p is odd.

Since degree of a point vertex v in $\overline{B_2}(G)$ is even, p – 1 + $deg_G(v)$ is even and hence $deg_G(v)$ is even, for all $v \in V(G)$

Case(ii): p is even.

Then p – 1 + $deg_G(v)$ is even implies that $deg_G(v)$ is odd, for all $v \in V(G)$. Hence, each vertex in G is of same parity.

Conversely, assume q is odd and degree of each vertex in G is of same parity. Then degree of each vertex in L(G) is even and hence degree of a line vertex e' in $\overline{B_2}(G)$ is q + 1 - $deg_{L(G)}(e')$ is even. Let v be a point vertex in $\overline{B_2}(G)$. If $deg_G(v)$ is odd, for all v in G, then since the number of odd degree vertices is even, p is even and hence the degree of v in $\overline{B_2}(G)$ is p – 1 + $deg_G(v)$ is even. If $deg_G(v)$ is even for all v in G, since q is odd, p is also odd and hence degree of v in $\overline{B_2}(G)$ is even Thus, degree of a point vertex is even. Hence, $\overline{B_2}(G)$ is Eulerian.

Theorem 3.2: If $\overline{L}(G)$ is Hamiltonian, then $\overline{B_2}(G)$ is Hamiltonian.

Proof: Assume $\overline{L}(G)$ is Hamiltonian. Then there exists a Hamiltonian cycle, say e_1' $e_2'\ldots e_q'$ e_1' in $\overline{L}(G)$. Let v_1 and v_q be any two vertices in G, incident with the edges in G

corresponding to the vertices e_1' and e_q' in the Hamiltonian cycle respectively. In the above Hamiltonian cycle, place v_q, v_1 in between e_q' and e_1' and then place the remaining point vertices in between v_q and v_1. This is possible, since the sub graph of $\overline{B_2}(G)$ induced by all point vertices is complete. This will form a Hamiltonian cycle of $\overline{B_2}(G)$ and hence $\overline{B_2}(G)$ is Hamiltonian.

Theorem 3.3: Let G be any (p, q) graph with p > q. If $\Delta_e(G) \leq \delta(G) + 1$, then $\overline{B_2}(G)$ is Hamiltonian, where $\Delta_e(G)$ is the maximum degree of L(G).

Proof: This theorem is proved by finding the closure of $\overline{B_2}(G)$. Since the sub graph of $\overline{B_2}(G)$ induced by the point vertices is complete, any two point vertices in $\overline{B_2}(G)$ are adjacent. Since $\Delta_e(G) \leq \delta(G) + 1$, the sum of the degrees of any two nonadjacent point, line vertices in $\overline{B_2}(G)$ exceeds p + q – 1 and they can be made adjacent. Therefore, in the closure of $\overline{B_2}(G)$, any two point vertices are adjacent and any two point, line vertices are adjacent. Construct a path in the closure of $\overline{B_2}(G)$ on 2q vertices with the initial vertex, a point vertex and the terminal vertex, a line vertex and point, line vertices occurring alternately. Then place the remaining p - q point vertices in the above path since p > q. Hence, there exists a Hamiltonian cycle in the closure of $\overline{B_2}(G)$ and is Hamiltonian. Thus, $\overline{B_2}(G)$ is Hamiltonian.

In the following, the radius and diameter of $\overline{B_2}(G)$ are determined. For simplicity, $\overline{d_2}(u)$, $\overline{e_2}(v)$ and $\overline{d_2}(u, v)$ are used to denote the degree of a vertex u, the eccentricity of a vertex v and the distance between the vertices u and v in $\overline{B_2}(G)$ respectively.

Theorem 3.4: Let G be any graph not totally disconnected with at least three vertices. Then diam($\overline{B_2}(G)$) = 2.

Proof: Since any two point vertices in $\overline{B_2}(G)$ are adjacent, distance between any two point vertices is 1. Let e_1' and e_2' be any two line vertices in $\overline{B_2}(G)$ and e_1 and e_2 be the corresponding edges in G. Then

$$\overline{d_2}(e_1', e_2') = \begin{cases} 1, & \text{if } (e_1, e_2) \notin E(G); \\ 2, & \text{if } (e_1, e_2) \in E(G). \end{cases}$$

Let v, e' be a point, line vertex in $\overline{B_2}(G)$ respectively and e be the edge in G corresponding to e'. Then

$$\overline{d_2}(v, e') = \begin{cases} 1, & \text{if } v \in e; \\ 2, & \text{if } v \notin e. \end{cases}$$

Since G has at least 3 vertices and not totally disconnected, it follows that diameter of $\overline{B_2}(G)$ is 2.

Corollary 3.4.1: $\overline{B_2}(G)$ is bi-eccentric with radius 1 if and only if $G \cong K_{1,n} \cup mK_1$, $K_2 \cup tK_1$, for $n \geq 2$, $m \geq 0$ and $t \geq 1$.

Proof: Radius of $\overline{B_2}(G)$ is 1 if and only if there exists a vertex v in G such that each edge in G is incident with v. That is, $G \cong K_{1,n} \cup mK_1$, $K_2 \cup tK_1$, for $n \geq 2$, $m \geq 0$ and $t \geq 1$.

Corollary 3.4.2: Let G be a graph with at least three vertices and not totally disconnected. $\overline{B_2}(G)$ is complete if and only if $G \cong nK_1$ or K_2, for $n \geq 2$.

Corollary 3.4.3: If G is none of the graphs $K_{1,n} \cup mK_1$, $K_2 \cup tK_1$, nK_1 and K_2, for $n \geq 2$, $m \geq 0$ and $t \geq 1$, then $\overline{B_2}(G)$ is self-centered with radius 2.

In the following, point independence number, point covering number and chromatic number for $\overline{B_2}(G)$ are obtained.

Theorem 3.5: For any connected graph G, $\beta_0(\overline{B_2}(G)) = \Delta(G)$ or $\Delta(G) + 1$, where $\Delta(G)$ is the maximum degree of G.

Proof: Since $\overline{L}(G)$ is an induced sub graph of $\overline{B_2}(G)$ and any two point vertices in $\overline{B_2}(G)$ are adjacent, a maximum independent set in $\overline{L}(G)$ together with a point vertex is a maximum independent set in $\overline{B_2}(G)$. If $G \cong C_3$ or G contains at least $\Delta(G) + 2$ vertices, then $\beta_0(\overline{B_2}(G))) = \Delta(G) + 1$. Otherwise, $\beta_0(\overline{B_2}(G)) = \Delta(G)$.

Next, point independence number for $\overline{B_2}(G)$ is obtained, when G is a disconnected graph.

Theorem 3.6: Let G be any disconnected graph (not totally disconnected) with $\Delta(G) = 2$. If one of the components of G is C_3, then $\beta_0(\overline{B_2}(G)) = 4$.

Proof: Let G contain C_3 as one of its components. Then the set of line vertices corresponding to the edges in C_3 and a point vertex corresponding to a vertex in any other component is the maximum independent set in $\overline{B_2}(G)$. Hence, $\beta_0(\overline{B_2}(G)) = 4$.

Remark 3.1:

(i). If $G \cong C_3 \cup K_2$, then $\beta_0(\overline{B_2}(G)) = 4$.

(ii). If G is disconnected and if either $\Delta(G) = 2$ and none of the components is C_3 or if $\Delta(G) > 3$ or $\Delta(G) = 1$, then $\beta_0(\overline{B_2}(G)) = \Delta(G)$ or $\Delta(G) + 1$.

Theorem 3.7: For any connected graph G, $\alpha_0(\overline{B_2}(G)) = p + q - \Delta(G)$ or $p + q - \Delta(G) - 1$.

Proof: This follows from $\alpha_0(\overline{B_2}(G)) + \beta_0(\overline{B_2}(G)) = p + q$ and Theorem 3.5. Similarly, $\alpha_0(\overline{B_2}(G))$ can be obtained by using Theorem 3.6 and Theorem 2.1.

Theorem 3.8: Let G be any disconnected graph (not totally disconnected) with $\Delta(G) = 2$. If one of the components of G is C_3, then $\alpha_0(\overline{B_2}(G)) = p + q - 4$.

Next, the chromatic number χ of $\overline{B_2}(G)$ is determined.

Theorem 3.9: For any (p, q) graph with $p \geq 3$, $\chi(\overline{B_2}(G)) = p$.

Proof: The sub graph of $\overline{B_2}(G)$ induced by all the p point vertices is complete. Hence $\chi(\overline{B_2}(G)) \geq p$. It is to be noted that $V(\overline{L}(G))$ can be partitioned into at most p - 1 independent sets. Since $\overline{L}(G)$ is an induced sub graph of $\overline{B_2}(G)$ and any line vertex in $\overline{B_2}(G)$ is adjacent to exactly two point vertices, any p-coloring can be extended to $\overline{B_2}(G)$. Thus, $\chi(\overline{B_2}(G)) = p$.

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Odd and Even Weak Convex Critical Graph and Domatic partition of Graphs

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Abstract: *In a graph G = (V, E), a set $D \subset V$ is a weak convex dominating(WCD) set if each vertex of V-D is adjacent to at least one vertex in D and $d_{<D>}(u,v) = d_G(u,v)$ for any two vertices u, v in D. A weak convex domination set D is said to be odd(even) W.C.D set, if for any vertex $u \in V$-D, there exists $v \in D$ at odd(even) distance from u. The domination number $\gamma_{oc}(G)$ is the smallest order of a odd weak convex dominating set of G and the domination number $\gamma_{ec}(G)$ is the smallest order of a odd weak convex dominating set of G. In this paper we study the change in the behaviour of even weak convex domination number with respect to addition of edges in the respective graph and also the domatic partition of a graph with respect to even dominating sets of a graph..*

Keywords: *domination number, distance, eccentricity, radius, diameter, self-centered, neighbourhood, weak convex dominating set, odd weak convex dominating set, even weak convex dominating set.*

1. Introduction

Graphs discussed in this paper are undirected and simple. Unless otherwise stated the graphs which we consider are connected graphs only. For a graph G, let V(G) and E(G) denote its vertex and edge set respectively. The *degree* of a vertex v in a graph G is denoted by $\deg_G(v)$. The length of any shortest path between any two vertices u and v of a connected graph G is called the *distance between u and v* and is denoted by $d_G(u, v)$. The distance between two vertices in different components of a disconnected graph is defined to be ∞. For a connected graph G, the *eccentricity* $e_G(v) = \max\{d_G(u, v): u \in V(G)\}$. If there is no confusion, we simply use the notion $\deg(v)$, $d(u, v)$ and $e(v)$ to denote degree, distance and eccentricity respectively for the concerned graph. The minimum and maximum eccentricities are the *radius* and *diameter* of G, denoted r(G) and diam(G) respectively. When these two are equal, the graph is called *self-centered* graph with radius r, equivalently is *r self-centered*. A vertex u is said to be an *eccentric vertex* of v in a graph G, if $d(u, v) = e(v)$. In general, u is called an eccentric vertex, if it is an eccentric vertex of some vertex. For $v \in V(G)$, the *neighbourhood* $N_G(v)$ of v is the set of all vertices adjacent to v in G. The set $N_G[v] = N_G(v) \cup \{v\}$ is called the *closed neighbourhood* of v. We define

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the set $N_G[S] = \bigcup_{u \in S} N_G(u)$ -S as the open *neighbourhood* of a set $S \subseteq V(G)$. A set S of edges in a graph is said to be *independent* if no two of the edges in S are adjacent. An edge e = (u, v) is a *dominating edge* in a graph G if every vertex of G is adjacent to at least one of u and v.

The concept of domination in graphs was introduced by Ore. A set $D \subseteq V(G)$ is called *dominating set* of G if every vertex in V(G)-D is adjacent to some vertex in D. D is said to be a *minimal* dominating set if D-{v} is not a dominating set for any $v \in D$. The *domination number* γ(G) of G is the minimum cardinality of a dominating set. We call a set of vertices a γ-set if it is a dominating set with cardinality γ(G). Different types of dominating sets have been studied by imposing conditions on the dominating sets. A dominating set D is called *connected* (*independent*) dominating set if the induced subgraph <D> is connected (independent). D is called a *total dominating set* if every vertex in V(G) is adjacent to some vertex in D.

A cycle of D of a graph G is called a *dominating cycle* of G, if every vertex in V – D is adjacent to some vertex in D. A dominating set D of a graph G is called a *clique dominating set* of G if <D> is complete. A set D is called an *efficient dominating set* of G if every vertex in V – D is adjacent to exactly one vertex in D. A set $D \subseteq V$ is called a *global dominating set* if D is a dominating set in G and $\overline{G}$. A set D is called a *restrained dominating set* if every vertex in V(G)-D is adjacent to a vertex in D and another vertex in V(G)-D. A set D is a *weak convex dominating set* if each vertex of V-D is adjacent to at least one vertex in D and the distance between any two vertices u and v in the induced graph <D> is equal to that of those vertices u and v in G. By $\gamma_c, \gamma_i, \gamma_t, \gamma_o, \gamma_k, \gamma_e, \gamma_g, \gamma_r$ and γ_{wc}, we mean the minimum cardinality of a connected dominating set, independent dominating set, total dominating set, cycle dominating set, clique dominating set, efficient dominating set, global dominating set and restrained dominating set respectively.

In this paper we introduce a new dominating set called acyclic weak convex dominating set of a graph through which we studied the properties of the graph such as variation in radius and diameter of the graph. We find upper and lower bounds for the new domination number in terms of various already known parameters. Also we studied several interesting properties like Nordhaus-Gaddum type results relating the graph and its complement.

2. Odd and Even Weak Convex Critical Graph

Definition 2.1:

A graph is said to be k - E.W.C.D critical graph if $\gamma_{ec}(G+e) < \gamma_{ec}(G)$ and $\gamma_{ec}(G) = k$, for any edge $e \notin E(G)$.

A graph is said to be k - O.W.C.D critical graph if $\gamma_{oc}(G+e) < \gamma_{oc}(G)$ and $\gamma_{oc}(G) = k$, for any edge $e \notin E(G)$.

Proposition 2.1:

There cannot be any 2–E.W.C.D critical graph.

Proof:

Since the only graph for which $\gamma_{ec} = 1$ is K_1 and 2- E.W.C.D critical means if we join any two non-adjacent vertices will yield $\gamma_{ec} = 1$. Thus there cannot be any 2-E.W.C.D critical graph.

Proposition 2.2:

A graph G is 3-E.W.C.D critical $\Leftrightarrow$ any one of the following holds good for any non-adjacent vertices u and v

(i) $N(u) \cup N(v) = V(G) - \{u,v\}$ and $N[u] \cap N[v] = \phi$

(ii) There exists a vertex w such that $N(u) \cup N(w) = V(G) - \{v\}$ and $N(u) \cap N(w) = \phi$ (or) $N(v) \cup N(w) = V(G) - \{u\}$ and $N(v) \cap N(w) = \phi$.

Proposition 2.3:

Any two pendant vertices cannot have a same support in a k-E.W.C.D critical graph.

Proof:

Since, if we join those pendant vertices, it will not reduce the domination number of the graph.

Proposition 2.4:

Any 3-E.W.C.D critical graph has at the most 2 pendant vertices.

Proof:

Let G be a 3- E.W.C.D critical graph. Let x, y and z be the pendant vertices of G and u, v and w be the supports of them respectively. Then clearly {u, v, w} is the only dominating set of G. If we join x and y then still G + xy have a 3-dominating set which is either {u, v, w} or {x, u, w} or {y, v, w}. Thus $\gamma_{ec}(G + xy) = 3$ only, which is a contradiction to G is 3-E.W.C.D critical. Hence G has at most 2 pendant vertices only.

Proposition 2.5:

The diameter of 3-E.W.C.D critical graph is at most 3.

Proof:

Let G be a 3-E.W.C.D critical graph.

Let u and v be any two non-adjacent vertices of G. Then the graph G+uv has a two dominating set, which contains any one of the vertex u or v. Without loss of generality assume that {u, w} dominates G+uv. If u dominates $N_1(v)$, then d(u, v) = 2. If w dominates $N_1(v)$, then d(u, v)$\leq$3. Hence the proof.

Proposition 2.6:

Any 3-E.W.C.D critical graph is a block.

Proof:

Let G be a 3-E.W.C.D critical graph. Let u be a cut vertex of G.

Let $C_1,C_2,\ldots\ldots,C_n$ be the components of G – u. Clearly all the vertices of C_i's are not adjacent to u. Therefore, there exists a vertex v in some component say C_1, which is of distance greater than or equal to 2. Also we have no vertex of distance greater than 3 from u, otherwise distance between that vertex and any other vertex will become greater than 4, which is a contradiction to the previous proposition. Thus we have d(u,v) = 2 in G. Also we have no other vertex in $C_2,\ldots\ldots.C_n$ is of distance greater than or equal to 2 from u, otherwise distance between v and those vertices will become more than 4. Hence all the vertices of $C_2,\ldots, C_n$ are adjacent to u. Now each of the components forms a clique, otherwise if we join any two non-adjacent vertices of any of the components $C_2,\ldots, C_n$ they will not reduce the domination number, which is a contradiction to G is critical. Therefore, each of the components $C_2,\ldots, C_n$ forms a clique. Now if we join any two vertices each from one of the components $C_2,\ldots, C_n$ will not affect the domination. Hence $C_2,\ldots, C_n$ form a single component only. Therefore, we have only two components C_1 and C_2 in which $C_2\cup\{u\}$ form a clique.

Now join any two vertices $v\in C_1 \cap N_1(u)$ and $w\in C_2$. Either v or w must be in the 2-dominating set of G+uw. Clearly {vx/x$\neq$w} cannot be a dominating set for G+vw. If x $\neq$ u then vx cannot dominate C_2. Also, if uv dominate G+vw it can dominate G also. Clearly uw also cannot form an even weak convex dominating set for G+vw, since there exists a vertex in C_1, which is at distance 2 from u (by previous arguments) as well as from w also. Clearly, {v, w} also cannot form an even dominating set as they are both adjacent to u. Hence there exists no 2-dominating set exist for G+vw. Also any singleton vertex cannot form an even dominating set for G+vw. Hence there cannot be two component C_1 and C_2 in G. Hence u will not be a cut vertex. Thus, G is a block.

Proposition 2.7:

A graph G is k-weak convex domination critical graph if and only if G is k-odd weak convex domination critical graph.

3. Even Weak Convex Domatic Number

Definition:

An Even Weak Convex Domatic Number d_{ec} of a graph G is the maximum partition { V_1, V_2,…, V_n} of V(G) such that each V_i, $1 \leq i \leq n$ is an E.W.C.D. set of G.

Observations:

3.1 : $d_{ec}(K_p) = 1$.

3.2 : If $d_{ec}(G) \geq 2$, then G is a block

3.3 : For any tree T, $d_{ec}(T) = 1$.

3.4 : $d_{ec}(K_{m,n}) = \begin{cases} 1 & \text{if } m \text{ or } n = 0 \\ 2 & \text{otherwise} \end{cases}$

3.5 : $d_{ec}(C_n) = \begin{cases} 1 & \text{if } p \neq 4 \\ 2 & \text{if } p = 4 \end{cases}$

3.6 : For any 0-W.C.D graph G, $d_{ec}(G) = 1$.

3.7 : For any geodetic block of diameter 2, $d_{ec}(G) \leq 2$.

3.8 : For Petersen graph, $d_{ec} = 2$, for other geodetic blocks $d_{ec} = 1$.

Proposition 3.1:

For any graph G, $d_{ec}(G) \leq \delta + 1$.

Proposition 3.2:

$d_{ec}(G) = \delta + 1 \Leftrightarrow G = K_1$

Proof:

Let G be a graph on p vertices with $d_{ec}(G) = \delta + 1$.

Let v be a vertex with degree δ. Then the E.W.C.D set in the domatic partition containing v does not contain any vertex other than v. (otherwise degree of v must be increased to δ+1). This implies that v itself form a dominating set for G. Since a E.W.C.D set must contain at least two vertices, $G = K_1$.

Proposition 3.3:

For any graph G with diameter $d \geq 3$, $d_{ec} \leq [n/(d-1)]$

Proposition 3.4:

For any graph G with radius r, $d_{ec} \leq [n/(2r-2)]$

Proposition 3.5:

For any graph $G \neq K_1$, $d_{ec} \leq \delta$.

Proposition 3.6:

For any graph G, $d_{ec}(G) + d_{ec}(\overline{G}) \leq p + 1$.

Proposition 3.7:

$d_{ec}(G) + d_{ec}(\overline{G}) = p + 1 \Leftrightarrow G = K_1$.

Proposition 3.8:

For any graph $G \neq K_1$, $d_{ec}(G) + d_{ec}(\overline{G}) \leq p - 1$.

Proposition 3.9:

For any cycle C_n, $n \geq 5$, $d_{ec}(C_n) + d_{ec}(\overline{C_n}) = 2$

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Distance Closed Domination in Graph

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Abstract: *In a graph G=(V,E), a set $S \subset V(G)$ is a distance closed set of G if for each vertex $u \in S$ and for each $w \in V$-S, there exists at least one vertex $v \in S$ such that $d_{<S>}(u, v) = d_G(u, w)$. Also, a vertex subset D of V(G) is a dominating set of G if each vertex in V-D is adjacent to at least one vertex in D. In this paper, we define a new concept of domination called distance closed domination (D.C.D) and analyze some structural properties of graphs and extremal problems relating to the above concepts.*

Keywords: *domination number, distance, eccentricity, radius, diameter, self centered graph, neighborhood, induced sub graph, unique eccentric point graph, ciliates, distance closed dominating set.*

1. Introduction

Graphs discussed in this paper are undirected and simple. Unless otherwise stated the graphs which we consider are connected and simple graphs only. For a graph, let V(G) and E(G) denotes its vertex and edge set respectively. The *degree* of a vertex v in a graph G is denoted by $deg_G(v)$. The length of any shortest path between any two vertices u and v of a connected graph G is called the *distance between u and v* and it is denoted by $d_G(u, v)$. The distance between two vertices in different components of a disconnected graph is defined to be ∞. For a connected graph G, the *eccentricity* $e_G(v) = \max \{d_G(u,v): \forall u \in V(G)\}$. If there is no confusion, we simply use the notion deg (v), d(u, v) and e(v) to denote degree, distance and eccentricity respectively for the connected graph. The minimum and maximum eccentricities are the *radius* and *diameter* of G, denoted by r(G) and diam (G) respectively. If these two are equal in a graph, that graph is called *self-centered* graph with radius r and is called an *r-self centered* graph. Such graphs are 2-connected graphs. Some structural properties are studied in [2] and [3]. A vertex u is said to be an eccentric vertex of v in a graph G, if d (u, v)=e(v) in that graph. In general, u is called an *eccentric vertex*, if it is an eccentric vertex of some vertex. For $v \in V(G)$, the *neighborhood* $N_G(v)$ of v is the set of all vertices adjacent to v in G. The set $N_G[v] = N_G(v) \cup \{v\}$ is called the *closed neighborhood* of v. A set S of edges in a graph is said to be independent if no two of the edges in S are adjacent. An edge e=(u, v) is a *dominating edge* in a graph G if every vertex of G is adjacent to at least one of u and v.

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The concept of distance in graph plays a dominant role in the study of structural properties of graphs in various angles using related concept of eccentricity of vertices in graphs. The study of structural properties of graphs using distance and eccentricity started with the study of centre of tree and propagated in different directions in the study of structural properties of graphs such as unique eccentric point graphs, k- eccentric point graphs, self centered graphs, graphs realizing given eccentricity sequence, radius, diameter and eccentric critical graphs and Hamiltonian properties in iterated line graphs. The structural and eccentricity properties of various graph operations and iterated graph operations are given in references [4], [5], [8], [10], [12] and [14].

The concept of domination in graphs was introduced by Ore [13]. A set $D \subseteq V(G)$ is called dominating set of G if every vertex in V(G)-D is adjacent to some vertex in D. D is said to be a minimal dominating set if D-{v} is not a dominating set for any $v \in D$. The *domination number* $\gamma(G)$ of G is the minimum cardinality of dominating sets. We call a set of vertices a γ-set if it is a dominating set with cardinality $\gamma(G)$.Different types of dominating sets have been studied by imposing conditions on the dominating sets. The sub graph of a graph G whose vertex set is S and whose edge set is the set of those edges of G that have both ends in S is called the induced sub graph of G induced by S and is denoted by <S>. A dominating set D is called *connected* (*independent*) dominating set if the induced sub graph <D> is connected (independent). D is called a *total dominating set* if every vertex in V(G) is adjacent to some vertex in D. One important aspect of the concept of distance and eccentricity is the existence of polynomial time algorithm to analyze them. The concept of domination in graphs originated from the chess games theory and that paved the way to the development of the study of various domination parameters and its relation to various other graph parameters. The list of survey of domination theory papers are in [6], [7], [11], [15], [16] and [17].

The new concepts such as distance closed sets, distance preserving sub graphs, eccentricity preserving sub graphs, Super eccentric graph of a graph, pseudo geodetic graphs are introduced and structural properties of those graphs are studied in [9].

Janakiraman and Alphonse [1] introduced and studied the concept of weak convex dominating sets, which mixes the concept of dominating set and distance preserving set. In this paper, we introduce a new dominating set called distance closed dominating set of a graph through which we studied the properties of the graph. We find upper and lower bounds for the new domination number in terms of various already known parameters. Also, we studied several interesting properties like Nordhaus-Gaddum type results relating the graph and its complement.

2. Prior Results

The concept of distance closed set is defined and studied in the doctoral thesis of Janakiraman [9] and the concept of distance closed sets in graph theory is due to the related concept of ideals in ring theory in algebra. The ideals in a ring are defined with respect to the multiplicative closure property with the elements of that ring. Similarly, the distance closed set is defined with respect to the distance property between the distance closed set and the vertices of the graph. The distance closed set of a graph G is defined as follows:

Let S be a vertex subset of G. Then S is said to be *distance closed set* of G if for each vertex $u \in S$ and for each $w \in V-S$, there exists at least one vertex $v \in S$ such that $d_{<S>}(u, v) = d_G(u, w)$.The properties of the above set and related structure in graphs are in [9].

Theorem 2.1 [9]**:** A vertex subset D of G is said to be a distance closed set if and only if

(i) $e(u/<D>) \geq e(u/G)$ for $u \in D$;

(ii) Every non eccentric point of <D> is a cut vertex.

Theorem 2.2 [9]**:** A graph G, which is not an odd path, is distance closed if and only if

(i) G is a unique eccentric point graph and;

(ii) Every vertex with eccentricity at most d-1 is a cut vertex, where d is the diameter of G.

Theorem 2.3 [9]**:** A graph G is 0-distance closed graph if and only if G is one of the following

(i) G is P_{2n+1};

(ii) G is a ciliate.

3. Main Results

In this paper, we define a new domination parameter namely, distance closed domination as follows.

3.1 Distance Closed Dominating Sets in Graphs

Definition 3.1

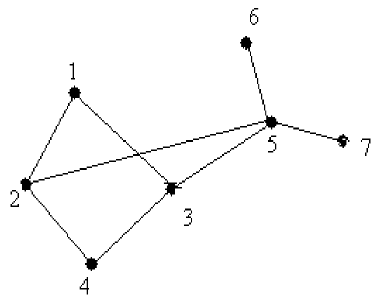
A subset $S \subseteq V(G)$ is said to be a distance closed dominating (D.C.D) set, if

(i) <S> is distance closed;

(ii) S is a dominating set.

The cardinality of a minimum D.C.D set of G is called the distance closed domination number of G and is denoted by γ_{dcl}.

Clearly, from the definition, $1 \le \gamma_{dcl} \le p$ and graph with $\gamma_{dcl}=p$ is called a 0-distance closed dominating graph. Also, if S is a D.C.D set of G then the complement V-S need not be a D.C.D set of G. For example



Here, S = { 1,3,5,6} form a D.C.D set but V-S ={2,4,7} is not a D.C.D set.

The following theorems give the bounds of some special class of graphs:

Theorem 3.1: If T is a tree with number of vertices $p \ge 2$, then $\gamma_{dcl}(T) = p-k+2$, where k is the number of pendant vertices in T.

Proof: The D.C.D set, S of a tree T must contain the diametrical eccentric points of T and also the induced sub graph < S > is a tree and it contains exactly the two pendant vertices of T. Hence $\gamma_{dcl}(T) = p-k+2$.

Theorem 3.2: Let G be a self centered graph of diameter 2.Then $\gamma_{dcl}(G) \le \delta+2$.

Proof: Any δ degree vertex, vertices in first neighborhood of that vertex and a vertex in the second neighborhood form a D.C.D set of G. Hence $\gamma_{dcl}(G) \le \delta+2$.

Theorem 3.3: Let G be a graph and S be a D.C.D set of G. Then the radius of the induced graph <S> is at least r, where r is the radius of G.

Proof: Let u be a point in the D.C.D set S with e(u)=r-1 in <S>. Then $e(u) \le r-1$ in G(from theorem 2.1).This implies that radius of G is at most r-1, a contradiction to radius of G is r. Hence, there is no point in the D.C.D set, which has eccentricity r-1 in the induced graph and hence the radius of <S> is at least r.

Corollary 3.1: Let G be a graph with radius r and diameter d. Let S be a D.C.D set of G. Then the diameter of the induced graph <S>, induced by S is at least r.

Proof: Proof follows from Theorem 2.1 and Theorem 3.3.

Theorem 3.4: Let G be a graph of order p. Then $\gamma_{dcl}(G)=2$ if and only if G has at least two vertices of degree p-1.

Proof: Assume that $\gamma_{dcl}(G)=2$. Let D= {u,v} be the distance closed dominating set of G. Then, we claim that d(u)=d(v)=p-1. As e(u)=e(v)=1 in <D>, e(u)≤1 and e(v)≤1 in G(from theorem 2.1).This implies that, e(u)=e(v)=1 in G. That is, d(u)=d(v)=p-1.

Conversely, let d(u)=d(v)=p-1in G. That is e(u)=e(v)=1.Then clearly {u,v} will form the distance closed dominating set of G. Hence $\gamma_{dcl}(G)=2$.

Proposition 3.1: If G is a (p,q) graph and $\gamma_{dcl}(G)=2$, then q≥ (4p-6)|2.

Proof: Let G be a (p,q) graph with $\gamma_{dcl}(G)=2$. Then any two vertices of degree p-1 of G belong to the distance closed dominating set D of G and d(u) ≥2 for all $u \in V-D$.

$$\begin{aligned}
\text{Therefore, } 2q &= \sum_{u \in V(G)} d(u) \\
&= \sum_{u \in D} d(u) + \sum_{u \in V-D} d(u) \\
&\geq 2(p-1) + (p-2)(2) \\
&= 2p-2+2p-4 \\
&= 4p-6
\end{aligned}$$

Hence, q ≥ (4p-6)|2.

Theorem 3.5: Let G be a graph of order p. If G has only one vertex of degree p-1, then $\gamma_{dcl}(G)=3$.

Proof: Let v be a vertex of degree p-1 in G. Then clearly, r(G)=1 and the diameter of G is 2. Hence, any two non adjacent vertices of eccentricity 2 together with v will form a distance closed dominating set of G and hence $\gamma_{dcl}(G)=3$.

Corollary 3.2: If G is a graph with $\gamma_{dcl}(G)=3$, then the diameter of G is 2.

Proof: Proof follows from Theorem 3.5.

Proposition 3.2: If G is a (p,q) graph and $\gamma_{dcl}(G)=3$, then q≥(p-1).

Proof: Let G be a (p,q) graph with $\gamma_{dcl}(G)=3$. Then G has exactly a vertex of degree p-1 and that vertex belongs to the distance closed dominating set D of G and d(u) ≥1 for all $u \in V-D$.

$$\begin{aligned}
\text{Therefore, } 2q &= \sum_{u \in V(G)} d(u) \\
&= \sum_{u \in D} d(u) + \sum_{u \in V-D} d(u) \\
&\geq 1(p-1) + (p-1)(1) \\
&= p-1+p-1 \\
&= 2p-2
\end{aligned}$$

Hence, $q \geq (p-1)$.

Theorem 3.6: If a graph G is connected and diam(G) ≥ 3, then $\gamma_{dcl}(\overline{G})=4$.

Proof: Since diam(G) ≥ 3, $\overline{G}$ has a dominating edge. Hence, $\gamma_{dcl}(\overline{G})=4$.

Nordhaus-Gaddum results for distance closed domination number:

Theorem 3.7: For any connected graph G such that $\overline{G}$ is also connected, $\gamma_{dcl}(G)+\gamma_{dcl}(\overline{G})\leq p+4$, where $\gamma_{dcl}(G)$ and $\gamma_{dcl}(\overline{G})$ are the cardinality of minimal distance closed dominating set of G and $\overline{G}$ respectively.

Proof: Let G be a connected graph G such that $\overline{G}$ is also connected.

Case 1: For r=1, d=1 or r=1, d=2.

Clearly, there is no graph G with the above cases such that both G and $\overline{G}$ are connected.

Case 2: For r=2 and d=2.

Consider a vertex v. Clearly $\{v\} \cup N_1(v) \cup$ a vertex in $N_2(v)$ forms a distance closed dominating set of G. In order to have both G and $\overline{G}$ are connected, there exists at least one vertex u in $N_1(u)$ having eccentric point in $N_2(u)$.In this case we have two sub cases.

Sub case(i): G has a dominating edge.

If G is a self centered graph of diameter 2 having a dominating edge, then clearly $\gamma_{dcl}(G)=4$ and $\overline{G}$ is of diameter≥ 3. Hence, $\gamma_{dcl}(\overline{G})\leq p$ and hence $\gamma_{dcl}(G)+\gamma_{dcl}(\overline{G})\leq p+4$.

Sub case(ii)): G has no dominating edge.

If G is a self centered graph of diameter 2 without dominating edge then $\overline{G}$ is also self centered graph of diameter 2.Then, $\gamma_{dcl}(G)\leq\delta(G)+2$ and $\gamma_{dcl}(\overline{G})\leq\delta(\overline{G})+2$(By theorem 3.2)

Therefore, $\gamma_{dcl}(G)+\gamma_{dcl}(\overline{G})\leq \delta(G)+2+\delta(\overline{G})+2 = \delta(G)+\Delta(G)+4 = p-1+4$

Hence $\gamma_{dcl}(G)+\gamma_{dcl}(\overline{G})\leq p+3$.

Case 3: For r≥2 and d≥3.

In G, there must exist at least two vertices, which has distance greater than or equal to 3. Then, clearly that two vertices form a dominating set for $\overline{G}$ and the eccentricity of that two vertices are 2. Therefore, $\gamma_{dcl}(\overline{G})=4$ whenever diam(G) ≥ 3.

Hence $\gamma_{dcl}(G)+\gamma_{dcl}(\overline{G})\leq p+4$.

Theorem 3.8: For any connected graph G such that $\overline{G}$ is also connected, $\gamma_{dcl}(G)+\gamma_{dcl}(\overline{G})= p+4$ if and only if one of the graphs G or $\overline{G}$ is a 0- distance closed dominating graph.

Proof: Let G be any graph. Assume that $\gamma_{dcl}(G)+\gamma_{dcl}(\overline{G})=p+4$. Then clearly, $\gamma_{dcl}(G)$ and $\gamma_{dcl}(\overline{G})$ are not less than or equal to 4. Thus either d(G) or $d(\overline{G})$ is greater than or equal to 3. Without loss of generality assume that d(G) is greater than or equal to 3.Then clearly, $\gamma_{dcl}(\overline{G})=4$ and hence $\gamma_{dcl}(G)+\gamma_{dcl}(\overline{G})=p+4$ implies that $\gamma_{dcl}(G)=p$, that is G is a 0-distance closed dominating graph.

Conversely, assume that G or $\overline{G}$ is a 0- distance closed dominating graph. Without loss of generality, let G be a 0- distance closed dominating graph. Then $\gamma_{dcl}(G)=p$ and $d(G)\geq 3$.This implies that $\gamma_{dcl}(\overline{G})=4$. Hence $\gamma_{dcl}(G)+\gamma_{dcl}(\overline{G})=p+4$.

Remark 3.1: The above bound is attainable for ciliates and paths.

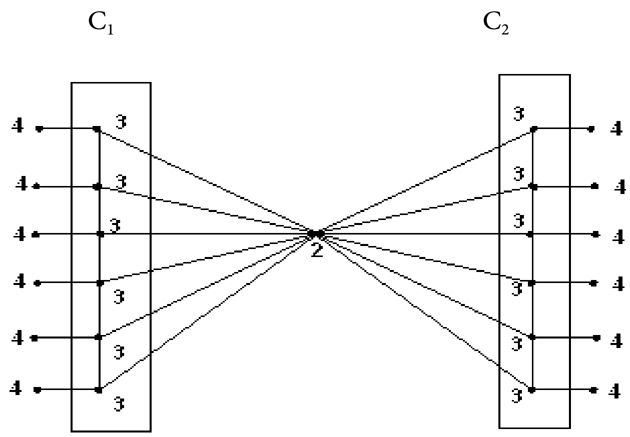
Theorem 3.9: There is no graph G such that both G and $\overline{G}$ are 0- distance closed dominating graphs.

Proof: Since all the 0-D.C.D graph are with diameter ≥ 3 and there is no graph for which both G and $\overline{G}$ are with diameter ≥ 3, we have the result.

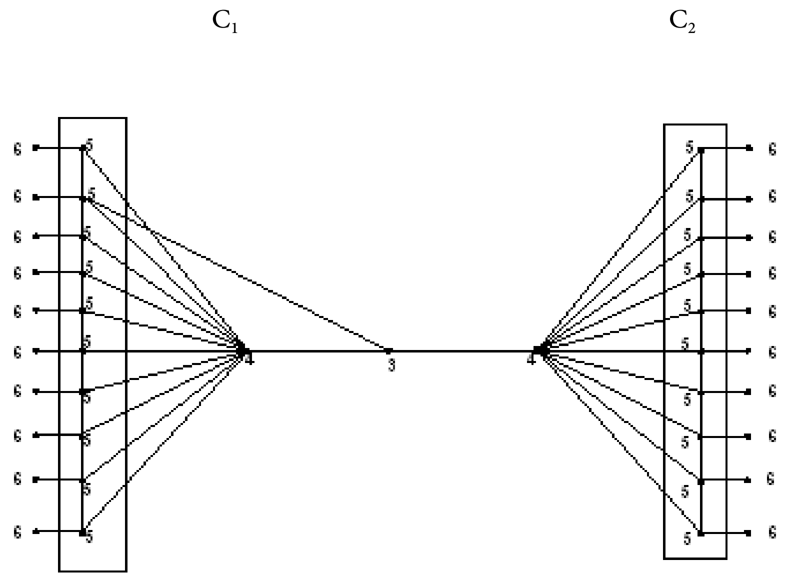
Remark 3.2:

The D.C.D set of a graph G need not be connected. We can see this in graphs with higher diameter. In particular, for any graph G with diameter≤3, the minimum D.C.D set must be connected. But graphs with diameter≥4 can have a disconnected minimum D.C.D set. For example, the following structure of graphs having a disconnected minimum D.C.D set.

Graphs having the following structure must have a disconnected minimum D.C.D set:



Graph with diameter 4 having a disconnected minimum D.C.D set $\{C_1, C_2\}$



Graph with diameter 6 having a disconnected minimum D.C.D set $\{C_1, C_2\}$

Similarly, we can construct graphs with diameter ≥7. Also these structure of graphs are having $\gamma=\gamma_{dcl}$.

Open problem:

1. For any graph G with diameter ≤3, the minimum D.C.D set must be connected-
   prove.
2. For every d ≥4, there exists at least one graph with diameter d, which has a disconnected minimum D.C.D set - prove.

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Eccentric Domatic Number of a Graph

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Abstract: *A subset D of the vertex set V(G) of a graph G is said to be a dominating set if every vertex not in D is adjacent to at least one vertex in D. A dominating set D is said to be an eccentric dominating set if for every $v \in V-D$, there exists at least one eccentric point of v in D. The minimum of the cardinalities of the eccentric dominating sets of G is called the eccentric domination number $\gamma_{ed}(G)$ of G. A partition of V(G) is called eccentric domatic if all its classes are eccentric dominating sets in G. The maximum number of classes of an eccentric domatic partition of V(G) is called the eccentric domatic number of G and is denoted by $d_{ed}(G)$. In this paper, bounds for $d_{ed}(G)$ and its exact value for some particular classes of graphs are studied.*

Key words: *Eccentric dominating set, Eccentric domination number, Eccentric domatic number.*

1.Introduction

Let G be a finite, simple, undirected graph on n vertices with vertex set V(G) and edge set E(G). For graph theoretic terminology and domination concepts refer to Harary [4], Buckley and Harary [1] and Haynes, Hedetniemi, and Slater [8].

Definition 1.1 Let G be a connected graph and u be a vertex of G. The **eccentricity** e(v) of v is the distance to a vertex farthest from v. Thus, $e(v) = \max \{d(u, v) : u \in V\}$. The **radius** r(G) is the minimum eccentricity of the vertices, whereas the **diameter** diam(G) is the maximum eccentricity. For any connected graph G, $r(G) \leq diam(G) \leq 2r(G)$. v is a central vertex if e(v) = r(G). The **center C(G)** is the set of all central vertices. The central subgraph $< C(G) >$ of a graph G is the subgraph induced by the center. v is a peripheral vertex if e(v) = d(G). The periphery P(G) is the set of all peripheral vertices.

For a vertex v, each vertex at a distance e(v) from v is an **eccentric vertex of v**. The set of all eccentric vertices of v is known as the **eccentric set E(v) of v**.

Definition 1.2 The **open neighborhood** N(u) of a vertex v is the set of all vertices adjacent to v in G. $N[v] = N(v) \cup \{v\}$ is called the **closed neighborhood** of v. For a vertex $v \in V(G)$, $N_i(u) = \{u \in V(G) : d(u, v) = i\}$ is defined to be the **i^{th} neighborhood** of v in G.

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Definition 1.3 [2,8] A set $S \subseteq V$ is said to be a **dominating set** in G, if every vertex in $V - S$ is adjacent to some vertex in S.

Definition 1.4 [3] A partition of V(G) is called **domatic** if all of its classes are dominating sets in G. The maximum number of classes of an domatic partition of V(G) is called the **domatic number** of G and is denoted by $d_d(G)$.

Definition: 1.5 [6] A set $D \subseteq V(G)$ is an **eccentric dominating set** if D is a dominating set of G and for every $v \in V - D$, there exists at least one eccentric point of v in D. If D is an eccentric dominating set, then every superset $D' \supseteq D$ is also an eccentric dominating set. But $D'' \subseteq D$ is not necessarily an eccentric dominating set.

An eccentric dominating set D is a **minimal eccentric dominating set** if no proper subset $D'' \subseteq D$ is an eccentric dominating set.

Definition: 1.6 [6] The **eccentric domination number $\gamma_{ed}(G)$** of a graph G equals the minimum cardinality of an eccentric dominating set. That is, $\gamma_{ed}(G) = \min |D|$, where the minimum is taken over D in D, where D is the set of all minimal eccentric dominating sets of G.

Obviously, $\gamma(G) \leq \gamma_{ed}(G)$.

Partitioning a given graph into sets such that each of which has some specified property has many applications in clustering a communication network. So in this section we define a new parameter known as **eccentric domatic number** of a given graph and study that parameter.

Definition: 1.6 A partition of V(G) is called **eccentric domatic** if all of its classes are eccentric dominating sets in G. The maximum number of classes of an eccentric domatic partition of V(G) is called the **eccentric domatic number of G** and is denoted by $d_{ed}(G)$.

For every graph G, there exists at least one eccentric partition of V(G), namely {V(G)}. Therefore, $d_{ed}(G)$ is well defined for every graph G. We give results on this parameter in section 2.

In [6], we have established the following results and are needed to study the eccentric domatic number of some classes of graphs.

Theorem:1.1 [6] $\gamma_{ed}(K_{,n}) = 1$.

Theorem:1.2 [6] If G is of diameter two $\gamma_{ed}(G) \leq 1+\delta(G)$.

Theorem: 1.3 [6] $\gamma_{ed}(P_n) = \lceil n/3 \rceil$, if n = 3k+1,

$$\gamma_{ed}(P_n) = \lceil n/3 \rceil + 1, \text{ if } n = 3k \text{ or } 3k+2.$$

Theorem: 1.4[6] (i) $\gamma_{ed}(C_n) = n/2$ if n is even.

(ii) $\gamma_{ed}(C_n) = \lceil n/3 \rceil$ or $\lceil n/3 \rceil + 1$, if n is odd.

Theorem: 1.5[6] $\gamma_{ed}(\overline{C_4}) = 2$, $\gamma_{ed}(\overline{C_5}) = 3$ and $\gamma_{ed}(\overline{C_n}) = \lceil n/3 \rceil$, $n \geq 6$.

2. Eccentric domatic number of a graph

First, we shall give some bounds for the eccentric domatic number of a graph.

Observation:2.1 If D is an eccentric dominating set in G, then for each $v \in V - D$, $D \cap N(v)$ and $D \cap E(v)$ are non empty sets. Hence we have, $d_{ed}(G) \leq 1 + \delta(G)$ and $d_{ed}(G) \leq \min_{v \in V(G)} \left(1 + |E(v)|\right)$.

Observation:2.2 If G is a graph on n vertices, then $1 \leq d_{ed}(G) \leq n$.

Now we give some observations, theorems and propositions relating eccentric domatic numbers of some classes of graphs.

Observation:2.3 For $n \geq 2$, $d_{ed}(K_{1,n}) = 1$.

Observation:2.4 For $n \geq 3$, $d_{ed}(P_n) = 1$.

Observation: 2.5 If G is a tree, $d_{ed}(G) \leq 2$.

Since every vertex of K_n is an eccentric dominating set the following proposition holds.

Proposition: 2.1 $d_{ed}(K_n) = n$.

In observation 2.2, both lower and upper bounds are sharp since $d_{ed}(K_n) = n$ and $d_{ed}(K_{1,n}) = 1$.

$d_{ed}(K_{m,n})$ is given by the following proposition.

Proposition: 2.2 For $2 \leq m \leq n$, $d_{ed}(K_{m,n}) = m$.

Proof: Let V_1, V_2 be the bipartition classes of $K_{m,n}$. Consider $u \in V_1$, $v \in V_2$. Clearly, {u,v} is a γ_{ed} dominating set.

Let $V_1 = \{u_1, u_2, \ldots, u_m\}$, $V_2 = \{v_1, v_2, \ldots, v_n\}$. Then $\{u_i, v_i\}$, $i = 1, 2, 3, \ldots, m-1$, $\{u_m, v_m, v_{m+1}, \ldots, v_n\}$ form an eccentric domatic partition of $K_{m,n}$. Hence $d_{ed}(K_{m,n}) \geq m$. On the other hand, $d_{ed}(K_{m,n}) \leq d(K_{m,n}) = m$. Therefore, $d_{ed}(K_{m,n}) = m$.

Proposition: 2.3 If $d_{ed}(G) \geq 3$, then $\delta(G) \geq 2$.

Proof: If $d_{ed}(G) \geq 3$, then every vertex of G has atleast two neighbours. Thus $\delta(G) \geq 2$.

Now, we will prove some results related to unique eccentric point graphs.

If G is an unique eccentric point graph, no vertex of G has more than one eccentric vertex hence we have

Proposition: 2.4 If G is an unique eccentric point graph then $d_{ed}(G) \leq 2$.

Proposition: 2.5 If G is an unique eccentric point graph such that each vertex is an eccentric point of a unique vertex with odd number of vertices then $d_{ed}(G) = 1$.

Proof: If G is an unique eccentric point graph, $\gamma_{ed}(G) \geq n/2$. Also G has an odd number of vertices. Hence $d_{ed}(G) = 1$.

Theorem 2.1 If G is an unique eccentric point graph with $\delta(G) = 1$, then $d_{ed}(G) = 1$.

Proof: If G is an unique eccentric point graph then $d_{ed}(G) \leq 2$ by Proposition 2.5. Let $u \in V(G)$ such that deg $u = 1$, and let v be the support of u in G. Since u is pendent, u and v have the same eccentric point w (unique).

Now, suppose that $d_{ed}(G) = 2$. Let $\{D_1, D_2\}$ be an eccentric domatic partition of G and let $u \in D_1$. Then $v \in D_2$ and $w \in D_2$. But $v \in D_2$ and D_1 is an eccentric dominating set implies that $w \in D_1$, which is a contradiction. Hence $d_{ed}(G) = 1$.

Theorem: 2.2 Let n be an even positive integer. Let G be obtained from the complete graph K_n by deleting edges of linear factor, then $d_{ed}(G) = 2$.

Proof: Let u and v be a pair of non-adjacent vertices in G. Then u and v are eccentric to each other. Also, G is a unique eccentric point graph and each vertex is an eccentric point of exactly one vertex. Therefore, $\gamma_{ed}(G) \geq n/2$. G is also regular self-centered with diameter 2. Consider, $D \subseteq V(G)$ such that $\langle D \rangle = K_{n/2}$. D contains n/2 vertices each vertex in $V - D$ is adjacent to atleast one element in D and each element in $V - D$ has its eccentric point in D. Hence $\gamma_{ed}(G) = n/2$. Also, there exists only two such partitions. Hence, $d_{ed}(G) = 2$.

In the following two theorems we study the number of domatic partitions of C_n and its complement.

Theorem: 2.3 (i) $d_{ed}(C_n) = 2$, if n is even.

(ii) $d_{ed}(C_n) = 2$, if n is odd and $n \neq 3m$.

(iii) $d_{ed}(C_n) = 3$, if $n = 3m$ and is odd.

Proof of (i): Let the cycle C_n be $v_1 v_2 v_3 \ldots v_{2k} v_1$. Each vertex of C_n has exactly one eccentric vertex and $\gamma_{ed}(C_n) = n/2$ if n is even. Hence $d_{ed}(C_n) \leq 2$.

If $n = 4$, any two adjacent vertices of C_4 is an eccentric dominating set of C_4.

Hence $d_{ed}(C_4) = 2$.

Let $n = 2k$ and $k > 2$.

case(i) k-odd.

Consider $D_1 = \{v_1, v_3, \ldots, v_k, v_{k+2}, \ldots, v_{2k-1}\}$ and $D_2 = \{v_2, v_4, \ldots, v_{k-1}, v_{k+1}, \ldots, v_{2k}\}$. This D_1 and D_2 is are eccentric dominating sets for C_n since they dominates C_n and v_i is an eccentric point of v_{i+k}. $\{D_1, D_2\}$ is an eccentric domatic partition of G. Hence $d_{ed}(C_n) = 2$.

case(ii) k even.

Let $D_1 = \{v_1, v_3, \ldots, v_{k-1}, v_{k+2}, v_{k+4}, \ldots, v_{2k}\}$. $D_2 = \{v_2, v_4, \ldots, v_k, v_{k+1}, v_{k+3}, \ldots, v_{2k-1}\}$. This D_1 is an eccentric dominating set for C_n since D_1 dominates C_n and v_i is an eccentric point of v_{i+k}. $\{D_1, D_2\}$ is an eccentric domatic partition of G. Hence $d_{ed}(C_n) = 2$.

Proof of (ii): Each vertex of C_n has exactly two eccentric vertices and two adjacent vertices and $\gamma_{ed}(C_n) = \lceil n/3 \rceil$ or $\lceil n/3 \rceil + 1$, if n is odd. Therefore, $d_{ed}(C_n) \leq 3$.

If $n = 2k+1$, $v_i \in V(G)$ has v_{i+k}, v_{i+k+1} as eccentric points.

case(i) $n = 3m+1$, n odd $\Rightarrow$ m is even.

Also $3m = 2k \Rightarrow$ k is a multiple of 3.

Consider $D = \{v_1, v_4, \ldots, v_{k+1}, v_{k+3}, v_{k+6}, \ldots, v_{2k-1}\}$. D is an eccentric dominating set and $|D| = \lceil n/3 \rceil = \gamma(C_n)$. Hence $\gamma_{ed}(C_n) = \lceil n/3 \rceil = m+1$. So $d_{ed}(C_n) \leq 2$. $V-D$ is also an eccentric dominating set. Therefore, $d_{ed}(C_n) = 2$.

case(ii) $n = 3m+2 \Rightarrow 3m$ is odd $\Rightarrow$ m is odd.

$$2k = 3m+1 = 3(m-1) + 4$$

$$k = 3l + 2$$

Consider $D = \{v_1, v_4, \ldots, v_{k-1}, v_k, v_{k+3}, \ldots, v_{2k+1}\}$. D is an eccentric dominating set with $\lceil n/3 \rceil + 1$ vertices and no γ-dominating set of C_n is an eccentric dominating set of C_n.

Hence $\gamma_{ed}(C_n) = \lceil n/3 \rceil + 1$ and as in the previous case $d_{ed}(C_n) = 2$.

Proof of (iii): When $n = 3m$, $\gamma_{ed}(C_n) = n/3 = \gamma(C_n)$. Each vertex of C_n has exactly two eccentric vertices and two adjacent vertices. Therefore, $d_{ed}(C_n) \leq 3$.

$n = 3m$, n odd $\Rightarrow$ m odd

$n = 3m = 2k+1 \Rightarrow 2k$ even and $2k = 3m-1$

$$2k = 3(m-1)+2$$

$$k = (3(m-1)+2)/2 \Longrightarrow k = 3l+1 \text{ (since } m-1 \text{ is even)}$$

Consider $D_1 = \{v_1, v_4, v_7, \ldots, v_k, v_{k+3}, v_{k+6}, \ldots, v_{2k-1}\}$,

$D_2 = \{v_2, v_5, v_8, \ldots, v_{k+1}, v_{k+4}, v_{k+7}, \ldots, v_{2k}\}$,

$D_3 = \{v_3, v_6, v_9, \ldots, v_{k+2}, v_{k+5}, v_{k+8}, \ldots, v_{2k+1}\}$.

Then D_1, D_2, D_3 form an eccentric domatic partition of $V(C_n)$. Hence, $d_{ed}(C_n) = 3$.

Theorem: 2.4 $d_{ed}(\overline{C_n}) = 2$ if $n \neq 3m$,

$d_{ed}(\overline{C_n}) = 3$ if $n = 3m$,

Proof: We know, $\gamma_{ed}(\overline{C_4}) = 2$, $\gamma_{ed}(\overline{C_5}) = 3$ and $\gamma_{ed}(\overline{C_n}) = \lceil n/3 \rceil$, $n \geq 6$ and each vertex of $\overline{C_n}$ has exactly two eccentric vertices when $n > 3$. Therefore, $d_{ed}(\overline{C_n}) \leq 3$.

Case 1: $n = 3m$

Let us assume that $v_1, v_2, \ldots, v_n, v_1$ form C_n. Then $D_1 = \{v_1, v_4, \ldots, v_{3m-2}\}$; $D_2 = \{v_2, v_5, \ldots, v_{3m-1}\}$; $D_3 = \{v_3, v_6, \ldots, v_{3m}\}$ form a partition of $V(\overline{C_n})$ into minimum eccentric dominating sets of $\overline{C_n}$. Hence $d_{ed}(\overline{C_n}) = 3$.

Case 2: $n \neq 3m$

In this case, $d_{ed}(\overline{C_n}) \leq 2$ since $\gamma_{ed}(\overline{C_n}) = \lceil n/3 \rceil > n/3$.

If $n = 3m+1$, $D = \{v_1, v_4, \ldots, v_{3m+1}\}$ and $V-D$ form an eccentric domatic partition of $\overline{C_n}$. Hence $d_{ed}(\overline{C_n}) = 2$.

If $n = 3m+2$, $D = \{v_1, v_4, \ldots, v_{3m+1}, v_{3m+2}\}$ and $V-D$ form an eccentric domatic partition of $\overline{C_n}$. Hence $d_{ed}(\overline{C_n}) = 2$.

Next theorem gives the eccentric domatic number of wheels.

Theorem: 2.5 $d_{ed}(W_3) = 4$, $d_{ed}(W_4) = 2$, $d_{ed}(W_5) = 2$, $d_{ed}(W_6) = 3$, $d_{ed}(W_7) = 2$ and $d_{ed}(W_n) = 3$, $n \geq 8$.

Proof: $W_3 \cong K_4$. Hence, $d_{ed}(W_3) = 4$.

$W_4 = K_1+C_4$. Let v_1, v_2, v_3, v_4 be the vertices of C_4 and v be the central vertex of W_4. Then $\{v, v_1, v_2\}$, $\{v_3, v_4\}$ are eccentric partitions of W_4. Hence $d_{ed}(W_4) = 2$.

Similarly, we can prove that $d_{ed}(W_5) = 2$, $d_{ed}(W_6) = 3$ and $d_{ed}(W_7) = 2$.

When $n \geq 8$, $W_n = K_1+C_n$.

Let $v_1, v_2, \ldots, v_n$ be the n vertices of C_n, then the central vertex v with any two vertices of C_n at distance three or more in C_n or v with any two adjacent vertices of C_n form an eccentric dominating set of W_n. Also, any dominating set of C_n is an eccentric dominating set of W_n. Hence, $d_{ed}(W_n) = d(C_n) = 3$ for $n \geq 8$.

A lower bound for eccentric domatic number is given in the following theorems.

Theorem: 2.6 If G is of radius greater than two, then $d_{ed}(G) \geq \lfloor n/(n-\delta(G) \rfloor$.

Proof: If G is a graph with radius greater than two, $V - N(u)$, where deg $u = \delta(G)$ is an eccentric dominating set. Let $D \subseteq V(G)$ with $|D| \geq n-\delta(G)$, then D is an eccentric dominating set. Thus we can take any $\lfloor n/(n-\delta(G)\rfloor$ disjoint subsets. Hence, $d_{ed}(G) \geq \lfloor n/(n-\delta(G)\rfloor$.

Theorem: 2.7 If G is self-centered of radius two, then $d_{ed}(G) \geq \lfloor n/(1+\Delta(G))\rfloor$.

Proof: If G is self-centered of radius two, N[u] is an eccentric dominating set for any $u \in V(G)$. Let $D \subseteq V(G)$ with $|D| \geq 1+\Delta(G)$, then D is an eccentric dominating set. Thus we can take any $\lfloor n/(1+\Delta(G))\rfloor$ disjoint subsets as eccentric dominating sets. Hence, $d_{ed}(G) \geq \lfloor n/(1+\Delta(G))\rfloor$.

Nordhaus-Gaddum type of results involving the domatic number of G and its complement is given in the following theorem.

Theorem: 2.8 $d_{ed}(G) + d_{ed}(\overline{G}) \leq n+1$ with equality if and only if $G = K_n$ or $\overline{K_n}$.

Proof: $d_{ed}(G) = 1$ and $d_{ed}(\overline{G}) = n$ when $G = K_n$ or $\overline{K_n}$. If $G \neq K_n$ or $\overline{K_n}$, $\gamma_{ed}(G) \geq 2$. Therefore $d_{ed}(G) \leq \lfloor n/2\rfloor$. Thus we see that $d_{ed}(G) + d_{ed}(\overline{G}) \leq n$.

Next, we proceed to prove eccentric domatic number of a graph or its complement is greater than two when the radius of G is greater than two.

Theorem: 2.9 Let G be a connected graph with radius greater than two. Then there exists a minimal eccentric dominating set D of G with the property that for $u \in D$, $N(u) \not\subseteq D$ and $V - N(u) \not\subseteq D$.

Proof: Let D be any minimal eccentric dominating set of G. For any $u \in V(G)$, $V - N(u)$ is an eccentric dominating set of G, but it is not minimal by theorem 2.5.. Hence $V - N(u) \not\subseteq D$. Next we prove that there exists D such that $N(u) \not\subseteq D$. Since G is of radius greater than two $D \neq N[u]$, since N[u] is not a dominating set of G and D must contain vertices which are at distance atleast two. Since D is minimal, N(u) is a subset of D if and only if each vertex of N(u) is the support of some pendent vertices. Let S_u be the set of all such pendent vertices. Take $D' = (D - N(u)) \cup S_u$. D' is a minimal eccentric dominating set of G (if x is a pendent vertex and y its support then x and y have same eccentric vertex) and $N(u) \not\subseteq D'$. Hence we can form a minimal dominating set D such that $N(u) \not\subseteq D$ for any $u \in D$. This proves the theorem.

Theorem: 2.10 Let G be a connected graph with radius $r > 2$. Then either G or $\overline{G}$ has atleast two disjoint eccentric dominating sets; that is $d_{ed}(G)$ or $d_{ed}(\overline{G})$ is ≥ 2.

Proof: When radius of G is greater than two, $\overline{G}$ is self-centered of diameter two. Hence in $\overline{G}$ two vertices are at distance two to each other implies that they are eccentric to each other. Hence for any $u \in V(G)$, $V-N(u)$ is an eccentric dominating set. Since radius of G is greater than two, vertices in N(u) has their eccentric vertices atleast at distance two from u. So, we can leave atleast one vertex v from $N_2(u)$ or from $N_3(u)$ such that $(V - N(u)) - \{v\}$ form an eccentric dominating set of G. Hence, $\gamma_{ed}(G) < n - \Delta(G)$. Thus we can have a minimal dominating set $D \subseteq V-N(u)$ of G. Let us prove that D and $V-D$ are eccentric dominating sets of $\overline{G}$.

Let $v \in V-D$. Since D is an eccentric dominating set of G, there exists $u, w \in D$ in G such that u is adjacent to v and w is eccentric to v in G. Therefore in $\overline{G}$, u is eccentric to v and w is adjacent to v. This proves that D is an eccentric dominating set of $\overline{G}$.

Now take $u \in D$. In G, u has some adjacent vertices in $V-D$ and some non-adjacent vertices in $V-D$. Hence in $\overline{G}$, u has some adjacent vertices in $V-D$ and some non-adjacent vertices that is eccentric vertices in $V-D$. This implies $V-D$ is also an eccentric dominating set of $\overline{G}$. Hence, $\overline{G}$ has atleast two disjoint eccentric dominating sets namely D and $V-D$. Therefore, $d_{ed}(\overline{G}) \geq 2$.

Corollary: 2.10 Let T be a tree with radius $r > 2$. Then $d_{ed}(\overline{T}) = 2$.

Theorem: 2.11 If G is a connected graph with radius > 2, then $2 \leq d_{ed}(G)\, d_{ed}(\overline{G}) \leq n^2/4$.

Proof: As in theorem 2.9, $d_{ed}(\overline{G}) \geq 2$. Thus, $2 \leq d_{ed}(G)\, d_{ed}(\overline{G}$ Now by theorem 2.7 $d_{ed}(G) + d_{ed}(\overline{G}) \leq n$. Thus $\sqrt{d_{ed}(G)\, d_{ed}(\overline{G})} \leq (d_{ed}(G) + d_{ed}(\overline{G}))/2 \leq n/2$; That is $d_{ed}(G)\, d_{ed}(\overline{G}) \leq n^2/4$.

Following theorem sharpened the bounds of theorem 2.8 for trees.

Theorem: 2.12 For any tree of order $n \geq 2$, $2 \leq d_{ed}(T) + d_{ed}(\overline{T}) \leq 4$. If radius of T is greater than two, then $3 \leq d_{ed}(T) + d_{ed}(\overline{T}) \leq 4$.

Proof: We know that $d_{ed}(T) \leq 2$. Also, since an end vertex of T has exactly one eccentric vertex in $\overline{T}$, $d_{ed}(\overline{T}) \leq 2$. Thus, $2 \leq d_{ed}(T) + d_{ed}(\overline{T}) \leq 4$. When the radius is greater than two, by theorem 2.9, $d_{ed}(\overline{T}) \geq 2$. So $d_{ed}(\overline{T}) = 2$. Thus we get $3 \leq d_{ed}(T) + d_{ed}(\overline{T}) \leq 4$.

The bounds in the previous theorem are sharp. When r = 2, $d_{ed}(T) + d_{ed}(\overline{T})=2$ for $T = P_4$. $d_{ed}(T) + d_{ed}(\overline{T}) = 4$ for a spider having more than three legs.

$d_{ed}(T) + d_{ed}(\overline{T})=3$ if $T = P_7$. $d_{ed}(T) + d_{ed}(\overline{T}) = 4$ if T is a tree as in figure 2.1.

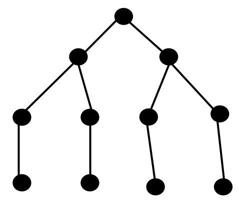

Figure 2.1

In the next theorem, we characterize trees T for which $d_{ed}(T) = 2$

Theorem: 2.13 Let T be a tree with diameter d, then $d_{ed}(T) = 2$ if and only if T satisfies the following conditions:

(i) P(T) has at least two pairs of peripheral vertices at distance d to each other.

(ii) Suppose (x, y), (z, w) are two such pairs, then support of x is different from supports of z and w and support of y is different from support of z and w.

Proof: Clearly $d_{ed}(T) \leq 2$.

Suppose T satisfies conditions (i) and (ii), let T_1 be the new graph obtained from T by removing the supports of x, y, z and w. Clearly T_1 is a tree and it has two disjoint dominating sets. Let them be D_1 and D_2. Consider $D_1' = D_1 \cup$ {supports of z and w} $\cup$ {end vertices adjacent to supports of x and y}. Clearly, x, y $\in D_1'$. Consider $D_2' = D_2 \cup$ {supports of x and y} $\cup$ {end vertices adjacent to supports of z and w}. Clearly, z, w $\in D_2'$. Then D_1', D_2' form an eccentric domatic partition of V(T). Thus $d_{ed}(T) = 2$.

On the other hand, assume that $d_{ed}(T) = 2$. Let $V(G) = D_1 \cup D_2$, where $\{D_1, D_2\}$ is an eccentric domatic partition of T. D_1 contains atleast two peripheral vertices at distance d to each other. Let them be x, y. D_2 contains atleast two peripheral vertices at distance d to each other. Let them be z, w. Therefore, T satisfies (i).

We know that x, y, z, w are end vertices of T. Since $\{D_1, D_2\}$ is an eccentric domatic partition of T, supports of x, y are in D_2 and supports of z, w are in D_1. Thus supports of x, y cannot be same as supports of z, w. This proves (ii). Hence the theorem is proved.

Now let us define eccentric domatically full graphs.

Definition 2.1 A graph G is **eccentric domatically full** if $d_{ed}(G) = 1+\delta(G)$.

Using this definition the previous theorem can be restated as follows.

Theorem: 2.14 Let T be a tree with diameter d. Then T is eccentric domatically full if and only if T satisfies the following conditions:

(i) P(T) has at least two pairs of peripheral vertices at distance d to each other.

(ii) Suppose (x, y), (z, w) are two such pairs, then support of x is different from supports of z and w and support of y is different from support of z and w.

Definition 2.2 A graph G is **eccentric domatic eccentrically full** if $d_{ed}(G) = \min_{v \in V(G)} \left(1 + |E(v)|\right)$.

Since for a tree, $\min_{v \in V(G)} |E(v)| = 1$, Theorem 2.13 characterizes trees which are eccentric domatic eccentrically full.

If T is a tree with radius r > 2 then from Corollary 2.10 it follows that $\overline{T}$ is always eccentric domatic eccentrically full

Also, a cycle C_n is eccentric domatically full if and only if n = 3m and n is odd since $d_{ed}(C_n) = 3$, if n = 3m and n is odd. C_n is eccentric domatic eccentrically full if and only if n = 3m and n is odd or n is even, since $d_{ed}(C_n) = 3$, when n = 3m and odd; and $d_{ed}(C_n) = 2$, when n is even.

Definition 2.2 A graph G is **domatically and eccentrically full** if $d_{ed}(G) = 1+\delta(G) = \min_{v \in V(G)} \left(1 + |E(v)|\right)$.

Thus trees satisfying (i) and (ii) in Theorem 2.13 are domatically and eccentrically full.

In the end, we prove an existing theorem.

Theorem: 2.15 Let V be a finite set with more than three vertices, and let k be any integer such that $1 \leq k \leq |V|/2$ and let $\{D_1, D_2, \ldots, D_k\}$ be a partition of V with $|D_i| \geq 2$. Then there exists a self-centered graph G with V(G) = V and $\{D_1, D_2, \ldots, D_k\}$ as an eccentric domatic partition.

Proof: In each D_i taking the elements as vertices, join each vertex to all other vertices by edges. Therefore, $\langle D_i \rangle$ is a complete graph for all i. Now, for any two distinct D_i, D_j split D_i into two parts X_{1i}, X_{2i} and D_j into two parts Y_{1j}, Y_{2j}. Join each vertex of X_{1i} to all the vertices of Y_{1j} and each vertex of X_{2i} to all vertices of Y_{2j}. But no vertex of X_{1i} is joined to vertices of Y_{2j} and no vertex of X_{2i} is joined to vertices of Y_{1j}. Name the new graph formed as G. Clearly G is self-centered of diameter 2. Also, for any i, D_i is an eccentric dominating set of G. Hence, $\{D_1, D_2, \ldots, D_k\}$ is an eccentric domatic partition of G and $d_{ed}(G) \geq k$. If $|D_i| = 2$ for all i, $d_{ed}(G)$ is exactly k since $\gamma_{ed}(G) \geq 2$ for $G \neq K_n$.

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Just Excellent Graphs

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Abstract: *A graph G is said to be excellent, if every vertex of G belongs to a γ - set. In this paper, we introduce a new class of graphs, called just excellent graphs and initiate a study on this class. A graph G is said to be just excellent if to each $u \in V$, there is a unique γ - set of G containing u. We obtain a necessary and sufficient condition for a graph to be just excellent. We find an upper bound for the domination number of a just excellent graph. If G is just excellent and γ (G) attains this upper bound, then we show that G is Hamiltonian. We show that every just excellent graph contains no cut vertex. We also prove that every graph is an induced subgraph of a just excellent graph.*

Key words: *Domination, Level vertex, Excellent graph, Just excellent graph.*

Introduction

We consider only simple undirected graphs. For graph theoretic terminologies we refer to [1]. Let $G = (V, E)$ be a graph. A set $D \subseteq V$ is a dominating set if every vertex in $V - D$ is adjacent to some vertex in D. The domination number $\gamma(G)$ of G is the minimum cardinality of a dominating set. A dominating set with cardinality $\gamma(G)$ is called γ - set of G. For information on domination refer to [2] and [3].

A graph G is said to be excellent, if every vertex of G belongs to a γ - set. The domatic number $d(G)$ of a graph G is defined to be the maximum number of elements in a partition of $V(G)$ into dominating sets.

Let $u \in V(G)$. Then $\gamma^u(G, u) = \min \{ |S| : S \subseteq V, S \text{ dominates } G - u \}$. By a $\gamma^u(G, u)$ - set we mean a set $S \subseteq V$, with $|S| = \gamma^u(S, G)$, which dominates $G - u$. For a vertex $u \in V(G)$

1. If $\gamma^u(G, u) = \gamma(G)$, then u is said to be a γ - level vertex of G, or simply a level vertex of G.
2. If $\gamma^u(G, u) = \gamma(G) - 1$, then u is said to be a γ - no level vertex of G, or simply a nonlevel vertex of G.

The private neighbor set of a vertex v in a γ - set S, denoted by $PN(v, S)$ is $N(v) - N[S - \{v\}]$ and each $u \in PN(v, S)$ is called the private neighbor of v with respect to S.

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Definition:

A graph G is said to be just excellent if to each $u \in V$, there is a unique γ - set of G containing u.

Remarks:

1. *Every just excellent graph is excellent.*
2. *If G is just excellent and $G \neq K_n$, there is no vertex u, such that $N[u]$ is a clique.*

   Proof:

   If there exist a vertex u such that $N[u]$ is complete, then consider the set γ – set S of G that contains u. Then $(S - u) \cup \{v\}$ is also a γ - set of G, for every $v \in N(u)$. Hence there are atleast two γ - sets of G containing elements of $S - u$, which is a contradiction.
3. *If G is just excellent, then* $\delta(u) \geq \frac{n}{\gamma(G)} - 1$.

   Proof:

   Let $V = S_1 \cup S_2 \cup \ldots \cup S_m$ be the partition of V into γ - sets of G. Fix one $u \in V$. Assume that $u \in S_j$. Since each S_i is a γ - set, u is adjacent to atleast one vertex of S_i, $i \neq j$. Hence $\delta(u) \geq m - 1 = \frac{n}{\gamma(G)} - 1$.
4. *If $G \neq K_2$, $\overline{K_n}$ is just excellent, then $\delta(G) \geq 2$. [In particular any tree $T \neq K_2$ is not just excellent.]*

   Proof:

   Assume that $G \neq K_2$, $\overline{K_n}$, and u be a pendant vertex of G. Let $N(u) = \{v\}$. Since G is just excellent, there exists a γ - set D of G containing u. As $u \in D$, $v \notin D$. As $G \neq K_2$ and $u \in D$, $|D| \geq 2$. So $(D-u) \cup \{v\}$ is a γ - set of G, which is a contradiction as G is just excellent.
5. *Every just excellent graph $G \neq \overline{K_n}$ is connected.*

   Proof:

   If G is not connected, (and as $G \neq \overline{K_n}$), one of the connected components G_1, of G contains more than one vertex. As G_1 is also just excellent, and $\gamma(G_1) \leq \frac{|G_1|}{2}$, G_1 has more than one γ - set. Select two γ - sets S_1 and S_2 of G_1. Fix one γ - set D for $G - G_1$. Then both $D \cup S_1$, and $D \cup S_2$ are γ - sets of G, which is a contradiction as G is just excellent. Hence every just excellent graph is connected.

6. *If $G \neq \overline{K_n}$ is just excellent, then $|\ PN\ (\ u, D\ )\ | \geq 2$ for all $u \in D$, where D is any γ- set of G.*

   Proof:

   Let D be a γ- set of G. If $PN(\ u, D\ ) = \phi$, $(\ D - u\ ) \cup \{\ w\ \}$ is also a γ- set, for any $w \in N(u)$. If $|\ PN(\ u, D\ )\ | = 1$, let $PN(\ u, D\ ) = \{w\}$. Then $(D - u) \cup \{w\}$ is a γ- set of G. In either case, we get a contradiction as G is just excellent. So $|\ PN(\ u,D)\ | \geq 2$, $\forall\ u \in D$.

7. *If G, $G \neq \overline{K_n}$ is just excellent, then $\Delta\ (\ G\ ) \leq\ n - 2k + 1$, where $k = \gamma(\ G\ )$.*

   Proof:

   Let $u \in V\ (G)$. Let S be a γ- set for G which contains u. $|PN(w, S)| \geq 2$, $\forall\ w \in S$. So u is not adjacent to any of the vertices in $\bigcup\limits_{w \neq u \in S} PN(w,S)$ and $deg(\ u\ ) \leq (\ n - 1\ ) - 2(\ |\ S\ |\ -\ 1\ )$. As this is true for all $u \in V\ (G)$, $\Delta \leq\ n-2k+1$.

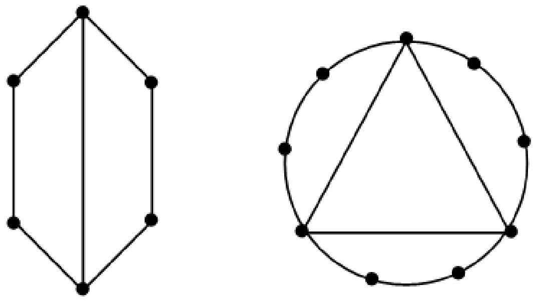


Fig. 1 Examples of graphs for which $\Delta\ =\ n - 2k\ + 1$.

Examples of graphs which are just excellent

1. Every cycle C_{3n} is just excellent.
2. Every complete graph K_n is just excellent.
3. Given a graph G with $\delta \geq 1$, a graph denoted by G_I is obtained as follows. To each $u \in V\ (\ G\ )$, a clique A_u of order $deg_G(\ u\ )$ is obtained, and a bijection $\phi_u : N(\ u\ ) \rightarrow A_u$ is constructed. $\phi_u(\ v\ )$ is denoted by v', $\forall\ v \in N(\ u\ )$.

   Now $V(G_I) = \bigcup E(A_u) \cup \{u'v' | uv \in E(G), v' \in A_u, u' \in A_v\}$

   Then $|\ V\ (\ G_I)\ | = 2\ |\ E\ (\ G\ )\ |$ and $|\ E(G_I)| = \frac{1}{2}\sum\limits_{u \in V}(\deg(u))^2$. The graph G_I is known as the inflated graph of G. G_I is just excellent when $G = K_n$ or C_{3n}. Also $\gamma(\ G_I) = n - 1$ if $G = K_n$, $(\ n \geq 2\ )$ and $\gamma(\ G_I) = 2n$ if $G = C_{3n}$.

4. I_n. Let $O_n = u_0, u_1, \ldots, u_{n-1}, u_0$ and $I_n = u_0', u_1', \ldots, u_{n-1}', u_0'$. We combine the cycles O_n and I_n and obtain a graph denoted by Y_n, where $E(Y_n) = \{u_i u_i', u_i u_{i+1}' \mid 0 \le i \le n-1\} \cup E(O_n) \cup E(I_n)$. The graphs Y_5, Y_{10}, Y_{15}, ... are just excellent.

In the following theorem, we obtain a necessary and sufficient condition for a graph to be just excellent.

Theorem 1:

The graph G is just excellent if and only if

1. $\gamma(G)$ *divides n.*
2. $d(G) = \dfrac{n}{\gamma(G)}$
3. *G has exactly* $\dfrac{n}{\gamma(G)}$ *distinct* γ*- sets.*

Proof:

Let G be just excellent. Let $S_1, S_2, \ldots, S_m$ be the collection of distinct γ- sets of G. Since G is just excellent, these sets are pair wise disjoint and their union is $V(G)$. So $V = S_1 \cup S_2 \cup \ldots \cup S_m$ is a partition of V into γ- sets of G. Since $|S_i| = \gamma(G)$, $\forall\ i = 1, 2, \ldots, m$ we have

1. domatic number of $G = m$, and
2. $m\gamma(G) = n$.

So both $\gamma(G)$ and $d(G)$ are divisors of n and $d(G) = \dfrac{n}{\gamma(G)}$. Also, G has exactly $m = \dfrac{n}{\gamma(G)}$ distinct γ- sets. Conversely, assume G to be a graph satisfying the hypothesis of the theorem. Let $m = \dfrac{n}{\gamma(G)}$. Let $V = S_1 \cup S_2 \cup \ldots \cup S_m$ be a decomposition of dominating sets of G. Now as $\gamma(G)m = n = \sum_{i=1}^{m} |S_i| \ge m\gamma(G)$, for each i, S_i is a γ- set of G. Since it is given that G has exactly m distinct γ- sets, $S_1, S_2, \ldots, S_m$ are the distinct γ- sets of G. Since $V = S_1 \cup S_2 \cup \ldots \cup S_m$ is a partition, each vertex of V belongs to exactly one S_i. Hence G is just excellent.

Theorem 2:

Let $G \neq K_2$ *be just excellent. Then* $\gamma(G) \le \dfrac{n}{3}$.

Proof:

Let D be a γ - set of G. Then by remark 6, $|\ PN(\ u,\ D\ )\ | \geq 2$, $\forall\ u\ \in D$. If $G\ \neq\ K_n$ is just excellent, then $d(\ G\ ) \geq 2$. If possible assume that $d\ (\ G\ )\ =\ 2$. Then $V = S_1 \cup\ S_2$, where S_1 and S_2 are the distinct γ - sets of G. As $|\ PN\ (u,\ S_1\ )\ |\ \geq 2$, $\forall\ u \in S_1$ and as $PN(\ u,\ S_1\ ) \subseteq S_2$ we get that $|\ S_2\ | \geq 2|\ S_1\ |\ =\ 2\gamma(\ G\ )$. But $|\ S_1\ |\ =\ |\ S_2\ |$. Hence $d(G) \geq 3$. Since G is just excellent, $n\ =\ \gamma(\ G\ )\ d(\ G\ )$. As $d(\ G\ ) \geq 3$, we get that $\gamma(G) \leq \frac{n}{3}$.

Remark:

The bound in the above theorem is sharp since $\gamma(\ C_{3n}\ ) = n$, $\forall\ n \geq 1$.

We now prove that the just excellent graphs for which the upper bound for $\gamma(\ G\ )$ is attained are Hamiltonian.

Theorem 3:

If G is just excellent and $\gamma(G) = \frac{n}{3}$, *then G is Hamiltonian.*

Proof:

If $\gamma(G) = \frac{n}{3}$ and G is just excellent, the domatic number $d\ (\ G\ )\ =\ 3$. Let S_1, S_2, S_3 be the distinct γ - sets of G. Then $V\ (\ G\ ) = S_1 \cup S_2 \cup S_3$. Let $A = \{e\ \in\ E\ (\ G\ )\ |\ e \in <S_i>\ \}$ for some $i\ =\ 1,\ 2,\ 3\}$. Then if $H\ =\ G\ -\ A$, then $\gamma(\ G\ )\ \leq\ \gamma(\ H\ )$. As S_1, S_2, S_3 are the dominating sets of H, $\gamma(\ G\ ) = \gamma(\ H\ )$. The sets S_1, S_2, S_3 are the distinct γ - sets of H, and H is just excellent. If H is Hamiltonian, then G is so. Hence it is enough to prove the result by assuming that each S_i is an independent set in G.

By remark 6, if $u\ \in S_i$, then $|\ PN(\ u,\ S_i\ )\ | \geq 2$ and as $u \neq v\ \in S_i \Longrightarrow PN(\ u,\ S_i\ )$ and $PN(\ v,\ S_i\ )$ are disjoint, it follows that $\bigcup_{u \in S_i} PN(u, S_i) \geq 2\gamma(G) = |V - S_i|$. [Since $|\ V\ -\ S_i\ |\ =\ n\ -\ \gamma(G)\ =\ 3\gamma(G)\ -\ \gamma(G)\ =\ 2\gamma(G)$]. Hence $deg\ (u) = 2$, $\forall u \in\ V\ (G)$. As $\gamma(G) = \frac{n}{3}$, $G \neq \overline{K_n}$, and as G is just excellent, G is connected. Since G is connected and 2 - regular, it is a cycle C_n.

Theorem 4:

Every just excellent graph G, contains no cut vertex and hence if $G \neq K_2$, *it contains no bridge.*

Proof:

If possible assume that G has a cut vertex u. Let H_1 be a component of $G\ -\ u$. Let G_1 be the induced subgraph $<\ H_1\ \cup\ \{u\}\ >$ of G and let $G_2\ =\ G\ -\ H_1$. Then clearly

$\gamma(G) \leq \gamma(G_1) + \gamma(G_2)$. Let S be the γ- set of G that contains u. Let D be a γ- set of G not containing u. [As $G \neq \overline{K_n}$ and G is just excellent, there is a γ- set of G not containing u]. Let $S_i = S \cap V(G_i)$ and $D_i = D \cap V(G_i)$, $i = 1, 2$. Since S_i dominates G_i, $|S_i| \geq \gamma(G_i)$. Now, $\gamma(G) = |S| = |S_1| + |S_2| - 1 \geq \gamma(G_1) + \gamma(G_2) - 1$.

Thus, $\gamma(G_1) + \gamma(G_2) - 1 \leq \gamma(G) \leq \gamma(G_1) + \gamma(G_2)$ ... (1).

Assume that $\gamma(G) = \gamma(G_1) + \gamma(G_2)$. Then whenever A and B are γ- sets of G_1 and G_2 respectively, $A \cup B$ is a dominating set of G. So $\gamma(G) \leq |A \cup B| \leq |A| + |B| = \gamma(G_1) + \gamma(G_2)$. As $\gamma(G) = \gamma(G_1) + \gamma(G_2)$, it follows that $A \cup B$ is a γ- set of G...(2). Clearly $u \notin A \cap B$, whenever A and B are γ- sets of G_1 and G_2 respectively. From (2) as G is just excellent, G_1 and G_2 have unique γ- sets. Atleast one of D_1 or D_2 dominates u. Assume that D_1 dominates u. Then D_1 is a γ- set of G_1. [If D_1 is not a γ- set of G_1, let A be a γ- set of G_1. Then $A \cup D_2$ is a dominating set of G and $|A \cup D_2| < |D_1 \cup D_2| = |D| = \gamma(G)$ a contradiction]. Either $|S_1| = \gamma(G_1)$ *or* $|S_2| = \gamma(G_2)$. [Otherwise $|S| = |S_1| + |S_2| - 1 \geq \gamma(G_1) + 1 + \gamma(G_2) + 1 - 1$, which is a contradiction]. Since D_1 is a γ- set of G_1 and $D_1 \neq S_1$, S_1 is not a γ-set of G_1. So S_2 is a γ- set of G_2. Now $D_1 \cup S_2$ and D are γ- sets of G, which is a contradiction as G is just excellent.

Thus, $\gamma(G) \neq \gamma(G_1) + \gamma(G_2)$ ... (3)

From the relation $|S| = \gamma(G) = \gamma(G_1) + \gamma(G_2) - 1$ and $|S_i| \geq \gamma(G_i)$ we get $|S_1| = \gamma(G_1)$ and $|S_2| = \gamma(G_2)$.

The vertex u cannot be a level vertex of both G_1 and G_2. [If not as D_i dominates $G_i - u$, $|D| = |D_1| + |D_2| = \gamma(G_1) + \gamma(G_2)$, which is a contradiction to (3)].

Also u cannot be a nonlevel vertex of both G_i. [If so select $A_i \subseteq V(G_i)$ such that A_i dominates $G_i - u$ and $|A_i| = \gamma(G_i) - 1$. Clearly $u \notin A_i$, and $A_1 \cup A_2 \cup \{w\}$ is a γ- set of G for any $w \in N[u]$ which is a contradiction as G is just excellent]. Let u be a level vertex of G_1 and a non level vertex of G_2. Let $B \subseteq V(G_2)$ such that B dominates $G_2 - u$, and $|B| = \gamma(G_2) - 1$. As $|S_1| = \gamma(G_1)$, S_1 is a γ- set of G_1. Since D_1 dominates $G_1 - u$ and u is a level vertex of G_1, $|D_1| \geq \gamma(G_1)$, $|D_2| \leq \gamma(G_2) - 1$ and D_2 is not a γ - set of G_2. Therefore D_1 dominates u and D_1 is a γ- set of G_1. Now $S_1 \cup B$ and $D_1 \cup B$ are γ- sets of G, which is a contradiction. In all cases we get a contradiction. Thus G has no cut vertex.

Theorem 5:

In a just excellent graph, $G \neq \overline{K_n}$ every vertex u is a level vertex, and also $\gamma(G - u) = \gamma(G)$.

Proof:

Let u be a vertex in G. Then there exist a γ - set of G not containing u. Hence $\gamma(G - u) \leq \gamma(G)$. We claim that $\gamma(G - u) = \gamma(G)$. If possible assume that $\gamma(G - u) < \gamma(G)$. So $G \neq K_n$. Let S be a γ - set for $G - u$. Then $S \cup \{ v \}$ is a γ - set for G, $\forall\ v \in N[u]$. As, G is connected, $N [u]$ contains atleast two vertices. So $S \cup \{ u \}$ and $S \cup \{ v \}$ are γ - sets for G for all $v \in N(u)$, a contradiction as G is just excellent. So $\gamma(G - u) = \gamma(G)$. If $\gamma^u(G, u) < \gamma(G)$, let S be a $\gamma^u(G, u)$ - set . If $u \in S$, then S is also a dominating set for G, which is a contradiction. If $u \notin S$, then S is a γ - set for $G - u$ and $\gamma(G - u) < \gamma(G)$ which is also a contradiction. Thus, $\gamma^u(G, u) = \gamma(G)$.

The converse of the above theorem need not be true. For example in C_{3n+2}, every vertex is a level vertex, but C_{3n+2} is not just excellent.

Theorem 6:

Let S_1, S_2, S_3 be the distinct γ - sets of C_{3n}. Then

1. *If $A \subset (E(\overline{C_{3n}})$ such that for every edge $e \in A$, e has both the end vertices in S_1, then $G = C_{3n} + A$ is just excellent. Further $d (G) = 3$.*
2. *If $u \in S_i$ and $v \in S_j$, $j \neq i$, then $C_{3n} + uv$ is not just excellent, where $uv \notin E (C_n)$.*

Proof:

Let $\{v_0, v_2, ..., v_{3n-1} \}$ be the vertices taken in clockwise order on the cycle C_{3n}. Without loss of generality let $S_i = \{ v_{3k+i} \mid k = 0, 1, ..., n - 1 \}$ where $i = 1, 2, 3$ (under addition modulo $3n$). Let D be a γ - set of G. If $D \cap S_1 = \phi$, then D is a γ - set of C_{3n} also. So in this case $\gamma(G) = \gamma(C_{3n}) = n$ and $D = S_2$ or S_3. If $D \cap S_1 \neq \phi$, we claim that $D = S_1$.

Assume that $v_{3t} \in D$, but $v_{3(t+1)} \notin D$. D must contain atleast one vertex from each of the subsets $\{ v_j \mid j = 3t + 1, 3t + 2 \}$, $\{ u_j \mid j = 3(t + k), 3(t + k) + 1, 3(t + k) + 2 \}$, where $k = 1, 2, ..., n - 1$ in order to dominate the vertices $v_{3(t + k)+1}$ and $v_{3t + 2}$. As D also contains v_{3t}, $| D | > n$ a contradiction as $\gamma(G) \leq \gamma(C_{3n}) = n$. So $v_{3t} \in D \Rightarrow v_{3(t + 1)} \in D$ and hence $S_1 \subseteq D$. As $\gamma(G) \leq \gamma(C_{3n})$, $D = S_1$. Then S_1, S_2, S_3 are the only γ - sets of G and hence G is just excellent and d (G) = 3.

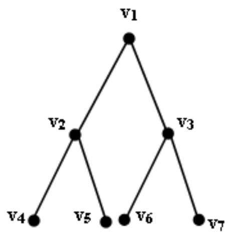
Let $u \in S_i$ and $v \in S_j$, i $\neq j$ and $uv \notin E(C_n)$. Let $j = i + 1 (\, mod\ 3\,)$. Let $u = v_{3t+1}$ and v = v_{3t+j}. Then S_j and $(S_j - v_{3t+j}) \cup \{v_{3t+i+2}\}$ are γ- sets of $G = C_n + uv$. Hence G is not just excellent.

Theorem 7:

Every graph is an induced subgraph of a just excellent graph.

Proof:

Let G be the given graph. If G is just excellent, then there is nothing to prove. Assume that G is not just excellent. Let $V(G) = \{v_1, v_2, ..., v_n\}$. Consider the cycle C_{3n}. It is just excellent. Let S_1, S_2, S_3 be the distinct γ- sets of C_{3n}. Label the vertices of S_1 by $u_1, u_2, \ldots, u_n$. Now in C_{3n} we add edges u_iu_j if and only if v_iv_j is an edge in G. Let the resulting graph be H. Then the induced subgraph $< S_1 >$ in H is isomorphic to G. By theorem 3, H is just excellent and $\gamma(H) = n$. Thus, the given graph G is an induced subgraph of a just excellent graph H. For example consider the tree in Fig. 2(a). It is not just excellent. This tree can be imbedded in a just excellent graph G as seen in Fig. 2(b).



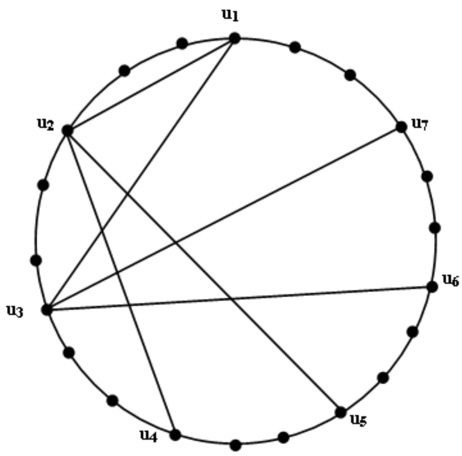


Fig. 2

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K – Even Mean Labeling of $D_{m,n}$ @ C_n

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Abstract: *Mean labeling of graphs was discussed in [10] and the concept of odd mean labeling was introduced in [9]. k-odd mean labeling and (k, d)-odd mean labeling are introduced and discussed in [5], [6], [7]. In this paper, we introduce the concept of k-even mean labeling and investigate k-even mean labeling of* $D_{m,n}$ @ C_n.

Keywords: *k-even mean labeling, k-even mean graph.*

AMS (MOS) Subject Classification: 05C78

1. Introduction

All graphs in this paper are finite, simple and undirected. Terms not defined here are used in the sense of Harary [8]. The symbols $V(G)$ and $E(G)$ will denote the vertex set and edge set of a graph. Labeled graphs serve as useful models for a broad range of applications [1-3].

A graph labeling is an assignment of integers to the vertices or edges or both subject to certain conditions. If the domain of the mapping is the set of vertices (or edges) then the labeling is called a vertex labeling (or an edge labeling).

Graph labeling was first introduced in the late 1960's. Many studies in graph labeling refer to Rosa's research in 1967 [11].

Labeled graphs serve as useful models for a broad range of applications such as X-ray crystallography, radar, coding theory, astronomy, circuit design and communication network addressing. Particularly interesting applications of graph labeling can be found in [4].

Mean labeling of graphs was discussed in [10] and the concept of odd mean labeling was introduced in [9]. k-odd mean labeling and (k, d) – odd mean labeling are introduced and discussed in [5], [6], [7]. In this paper, we introduce the concept of k-even mean labeling and here we investigate the k-even mean labeling of $D_{m,n}$ @ C_n.

Throughout this paper, k denotes any positive integer ≥ 1. For brevity, we use k-EML for k-even mean labeling.

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2. Main Results

2.1. Definition: *k*-even mean labeling

A (p, q) graph G is said to have a k-even mean labeling if there exists a injection $f: V \rightarrow \{0, 1, 2, ..., 2k + 2(q - 1)\}$ such that the induced map $f^* : E(G) \rightarrow \{2k, 2k + 2, 2k + 4, ..., 2k + 2(q - 1)\}$ defined by

$$f^*(uv) = \begin{cases} \dfrac{f(u)+f(v)}{2} & \text{if } f(u)+f(v) \text{ is even} \\ \dfrac{f(u)+f(v)+1}{2} & \text{if } f(u)+f(v) \text{ is odd} \end{cases} \text{ is a bijection.}$$

A graph that admits a k-even mean labeling is called a k-even mean graph.

2.2. Definition :

A dragon is formed by joining the end point of a path to a cycle. In fact, it is the one-point union of the end point of a path to a vertex of a cycle. Koh et al. call these tadpoles. Kim and Park call them kites [4].

Let $D_{m,n}$ denote the one-point union of the end point of the path P_m to a vertex of a cycle C_n and $D_{m,n}$ @ C_n denote the graph obtained by the one-point union of the end point of the dragon $D_{m,n}$ to a vertex of a cycle C_n.

2.3. Theorem

$D_{m,n}$ @ C_n, $n \equiv 0 \pmod 4$ is a k-even mean graph for any k and $m > 2$.

Proof

Let $V(D_{m,n} @ C_n) = \{v_i, 1 \leq i \leq n + m - 2\} \cup \{v_i', 1 \leq i \leq n\}$ and

$E(D_{m,n} @ C_n) = \{e_i, 1 \leq i \leq n + m - 1\} \cup \{e_i', 1 \leq i \leq n\}$ (see Fig. 2.1)

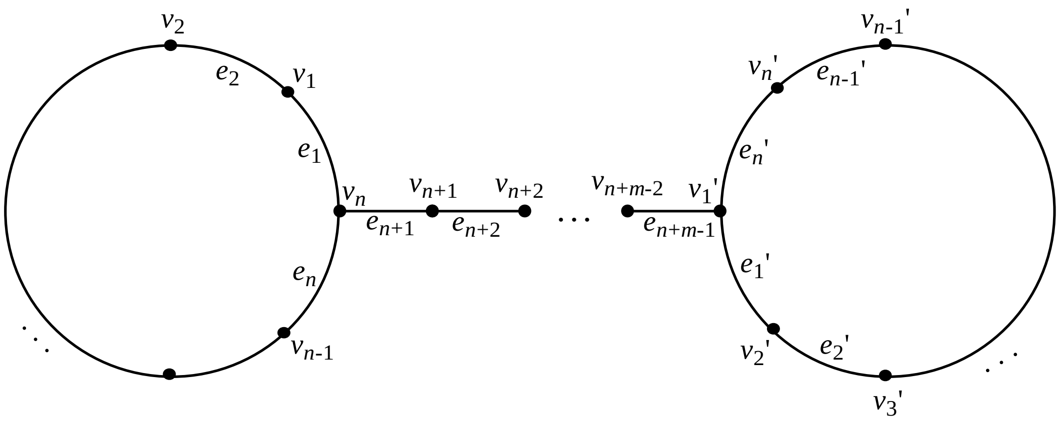

Fig. 2.1: Ordinary labeling of $D_{m,n}$ @ C_n

First we label the vertices of $D_{m,n}$ @ C_n as follows:

Define $f: V(D_{m,n} @ C_n) \rightarrow \{0, 1, 2, ..., 2k + 2q - 2\}$ by

For $1 \leq i \leq \frac{n-2}{2}$,

$$f(v_i) = \begin{cases} 2k+2n-4(i+1)+1 & \text{if } i \text{ is odd} \\ 2k+2n-4i-2 & \text{if } i \text{ is even} \end{cases}$$

For $\frac{n}{2} \leq i \leq n$,

$$f(v_i) = \begin{cases} 2k+4i-2n+1 & \text{if } i \text{ is odd} \\ 2k-2(n+1)+4i+1 & \text{if } i \text{ is even} \end{cases}$$

For $n + 1 \leq i \leq n + m - 2$,

$$f(v_i) = 2k + 2(i - 1) + 1$$

For $1 \leq i \leq \frac{n}{2}$,

$$f(v_i') = \begin{cases} 2k+2(n+m)+4(i-2)+1 & \text{if } i \text{ is odd} \\ 2k+2(n+m)-1+4(i-2) & \text{if } i \text{ is even} \end{cases}$$

For $\frac{n+2}{2} \leq i \leq n$,

$$f(v_i') = \begin{cases} 2k+2(3n+m)-4i & \text{if } i \text{ is odd} \\ 2k+2(3n+m)+3-4i & \text{if } i \text{ is even} \end{cases}$$

Then the induced edge labels are

$$f^*(e_i) = \begin{cases} 2k+2n-4i, & 1 \leq i \leq \frac{n}{2} \\ 2k+4i-2n-2, & \frac{n+2}{2} \leq i \leq n \end{cases}$$

For $n + 1 \leq i \leq n + m - 1$,

$$f^*(e_i) = 2k + 2i - 2$$

$$f^*(e_i') = \begin{cases} 2k+2(n+m)+4(i-1)-2, & 1 \leq i \leq \frac{n}{2} \\ 2k+2(3n+m)-4i, & \frac{n+2}{2} \leq i \leq n \end{cases}$$

Therefore, $f^*(E(D_{m,n} @ C_n)) = \{2k, 2k + 2, 2k + 4, \ldots, 2k + 2q - 2\}$. So, f is a k-even mean labeling and hence, $D_{m,n}$ @ C_n, $n \equiv 0 \pmod 4$) is a k-even mean graph for any k.

7-*EML* of $D_{4,8}$ @ C_8 is shown in Fig. 2.2.

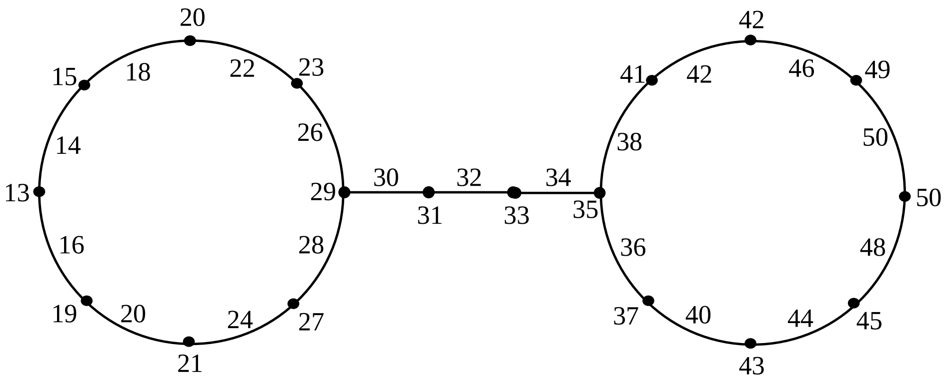

Fig. 2.2: 7-*EML* of $D_{4,8}$ @ C_8

2.4. Theorem

$D_{m,n}$ @ C_n, $n \equiv 1 \pmod 4$ is a k-even mean graph for any k and $m > 2$.

Proof

Let $V(D_{m,n} @ C_n) = \{v_i, 1 \leq i \leq n + m - 2\} \cup \{v_i', 1 \leq i \leq n\}$ and

$E(D_{m,n} @ C_n) = \{e_i, 1 \leq i \leq n + m - 1\} \cup \{e_i', 1 \leq i \leq n\}$ (see Fig. 2.1)

First we label the vertices of $D_{m,n}$ @ C_n as follows:

Define $f: V(D_{m,n} @ C_n) \longrightarrow \{0, 1, 2, ..., 2k + 2q - 2\}$ by

For $1 \leq i \leq \frac{n-3}{2}$,

$$f(v_i) = 2k + 2n - 4i - 2$$

For $\frac{n-1}{2} \leq i \leq n - 1$,

$$f(v_i) = \begin{cases} 2k + 4i - 2n - 1 & \text{if } i \text{ is odd} \\ 2k + 4i - 2n + 1 & \text{if } i \text{ is even} \end{cases}$$

$$f(v_n) = 2k + 2n - 2$$

For $n + 1 \leq i \leq n + m - 2$,

$$f(v_i) = 2k + 2i - 1$$

For $1 \leq i \leq \frac{n+1}{2}$,

$$f(v_i') = \begin{cases} 2k + 2(n+m) + 4(i-2) + 1 & \text{if } i \text{ is odd} \\ 2k + 2(n+m) + 4(i-2) - 1 & \text{if } i \text{ is even} \end{cases}$$

For $\frac{n+3}{2} \leq i \leq n$,

$$f(v_i') = 2k + 2(3n + m) - 4i + 2$$

Then the induced edge labels are

$$f^*(e_i) = \begin{cases} 2k + 2n - 4i, & 1 \leq i \leq \frac{n-1}{2} \\ 2k + 2(2i - n) - 2, & \frac{n+1}{2} \leq i \leq n \end{cases}$$

For $n + 1 \leq i \leq n + m - 1$,

$$f^*(e_i) = 2k + 2i - 2$$

$$f^*(e_i') = \begin{cases} 2k + 2(n+m) + 4(i-1) - 2, & 1 \leq i \leq \frac{n+1}{2} \\ 2k + 2(3n+m) - 4i, & \frac{n+3}{2} \leq i \leq n \end{cases}$$

Therefore, $f^*(E(D_{m,n} @ C_n)) = \{2k, 2k + 2, 2k + 4, ..., 2k + 2q - 2\}$. So, f is a k-even mean labeling and hence, $D_{m,n} @ C_n$, $n \equiv 1 \pmod 4$ is a k-even mean graph for any k.

7-*EML* of $D_{5,5}$ @ C_5 is shown in Fig. 2.3.

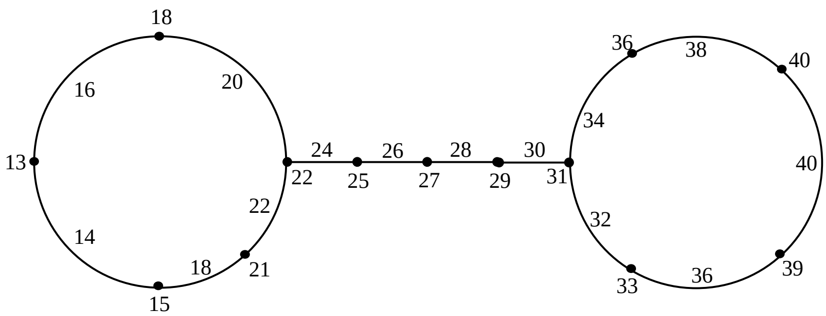

Fig. 2.3: 7-*EML* of $D_{5,5}$ @ C_5

3-*EML* of $D_{3,9}$ @ C_9 is shown in Fig. 2.4.

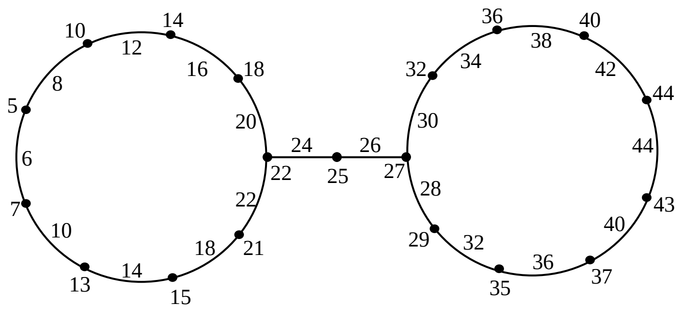

Fig. 2.4: 3-*EML* of $D_{3,9}$ @ C_9

2.5. Theorem

$D_{m,n}$ @ C_n, $n \equiv 2 \pmod 4$ is a k-even mean graph for any k when $n > 6$ and $m > 2$.

Proof

Let $V(D_{m,n} @ C_n) = \{v_i, 1 \leq i \leq n + m - 2\} \cup \{v_i', 1 \leq i \leq n\}$ and

$$E(D_{m,n} @ C_n) = \{e_i, 1 \leq i \leq n + m - 1\} \cup \{e_i', 1 \leq i \leq n\} \text{ (see Fig. 2.1)}$$

First we label the vertices of $D_{m,n} @ C_n$ as follows:

Define $f : V(D_{m,n} @ C_n) \rightarrow \{0, 1, 2, ..., 2k + 2q - 2\}$ by

$$f(v_1) = 2k + 2n - 3$$

For $2 \leq i \leq \frac{n-2}{2}$,

$$f(v_i) = 2k + 2(i - 3) + 1$$

$$f\left(v_{\frac{n}{2}}\right) = 2k + n - 6$$

$$f\left(v_{\frac{n+2}{2}}\right) = 2k + n - 2$$

For $\frac{n+4}{2} \leq i \leq n - 2$,

$$f(v_i) = 2k + 2(i - 2) + 1$$

$$f(v_{n-1}) = 2k + 2n - 6$$

$$f(v_n) = 2k + 2n - 2$$

For $n + 1 \leq i \leq n + m - 2$,

$$f(v_i) = 2k + 2i - 1$$

For $1 \leq i \leq \frac{n-4}{2}$,

$$f(v_i') = 2k + 2(n + m) + 2i - 5$$

$$f\left(v_{\frac{n-2}{2}}'\right) = 2k + 3n + 2m - 8$$

$$f\left(v_{\frac{n}{2}}'\right) = 2k + 3n + 2m - 4$$

For $\frac{n+2}{2} \leq i \leq n - 3$,

$$f(v_i') = 2k + 2(n + m) + 2(i - 2) + 1$$

$$f(v_{n-2}') = 2k + 2(2n + m) - 8$$

$$f(v_{n-1}') = 2k + 2(2n + m) - 4$$

$$f(v_n') = 2k + 2(2n + m) - 5$$

Then the induced edge labels are

$$f^*(e_1) = 2k + 2n - 2$$

$$f^*(e_2) = 2k + n - 2$$

For $3 \leq i \leq \frac{n+2}{2}$,

$$f^*(e_i) = 2k + 2i - 6$$

For $\frac{n+4}{2} \leq i \leq n$,

$$f^*(e_i) = 2k + 2i - 4$$

For $n + 1 \leq i \leq n + m - 1$,

$$f^*(e_i) = 2k + 2i - 2$$

For $1 \leq i \leq \frac{n-2}{2}$,

$$f^*(e_i') = 2k + 2(n + m) + 2(i - 1) - 2$$

For $\frac{n}{2} \leq i \leq n - 1$,

$$f^*(e_i') = 2k + 2(n + m) + 2i - 2$$

$$f^*(e_n') = 2k + 3n + 2m - 4$$

Therefore, $f^*(E(D_{m,n} @ C_n) = \{2k, 2k + 2, 2k + 4, ..., 2k + 2q - 2\}$. So, f is a k-even mean labeling and hence, $D_{m,n} @ C_n$, $n \equiv 2 \pmod 4$ is a k-even mean graph for any k when $n > 6$.

6-*EML* of $D_{4,10}$ @ C_{10} is shown in Fig. 2.5.

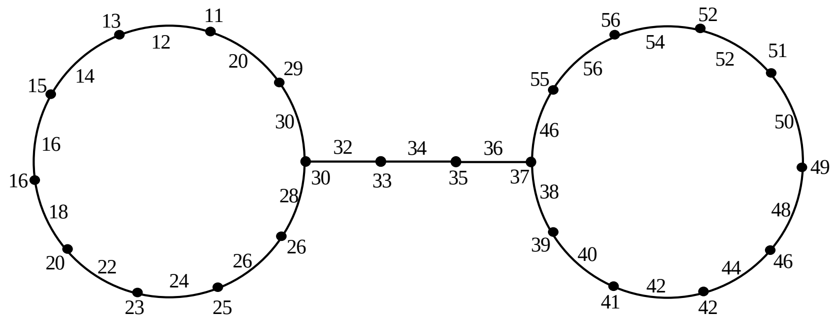

Fig. 2.5: 6-*EML* of $D_{4,10}$ @ C_{10}

2.6. Theorem

$D_{m,n}$ @ C_n, $n \equiv 3 \pmod 4$ is a k-even mean graph for any k and $m > 2$.

Proof

Let $V(D_{m,n} @ C_n) = \{v_i, 1 \leq i \leq n + m - 2\} \cup \{v_i', 1 \leq i \leq n\}$ and

$E(D_{m,n} @ C_n) = \{e_i, 1 \leq i \leq n + m - 1\} \cup \{e_i', 1 \leq i \leq n\}$ (see Fig. 2.1)

First we label the vertices of $D_{m,n} @ C_n$ as follows:

Define $f: V(D_{m,n} @ C_n) \longrightarrow \{0, 1, 2, ..., 2k + 2q - 2\}$ by

For $1 \leq i \leq \frac{n-3}{2}$,

$$f(v_i) = 2k + 2i - 3$$

$$f\left(v_{\frac{n-1}{2}}\right) = 2k + n - 5$$

$$f\left(v_{\frac{n+1}{2}}\right) = 2k + n - 1$$

For $\frac{n+3}{2} \leq i \leq n + m - 2$,

$$f(v_i) = 2k + 2i - 1$$

For $1 \leq i \leq \frac{n-3}{2}$,

$$f(v_i') = 2k + 2(n + m) + 2i - 5$$

$$f\left(v_{\frac{n-1}{2}}'\right) = 2k + 3n + 2m - 7$$

$$f\left(v_{\frac{n+1}{2}}'\right) = 2k + 3n + 2m - 3$$

For $\frac{n+3}{2} \leq i \leq n - 1$,

$$f(v_i') = 2k + 2(n + m) + 2(i - 2) + 1$$

$$f(v_n') = 2k + 2(2n + m) - 4$$

Then the induced edge labels are

$$f^*(e_1) = 2k + n - 1$$

$$f^*(e_i) = \begin{cases} 2k + 2i - 4, & 2 \leq i \leq \frac{n+1}{2} \\ 2k + 2i - 2, & \frac{n+3}{2} \leq i \leq n + m - 1 \end{cases}$$

$$f^*(e_i') = \begin{cases} 2k + 2(n+m) + 2i - 4, & 1 \leq i \leq \frac{n-1}{2} \\ 2k + 2(n+m) + 2i - 2, & \frac{n+1}{2} \leq i \leq n - 1 \end{cases}$$

$$f^*(e_n') = 2k + 3n + 2m - 2$$

Therefore, $f^*(E(D_{m,n} @ C_n)) = \{2k, 2k + 2, 2k + 4, ..., 2k + 2q - 2\}$. So, f is a k-even mean labeling and hence, $D_{m,n}$ @ C_n, $n \equiv 3 \pmod 4$ is a k-even mean graph for any k.

2-*EML* of $D_{3,11}$ @ C_{11} is shown in Fig. 2.6.

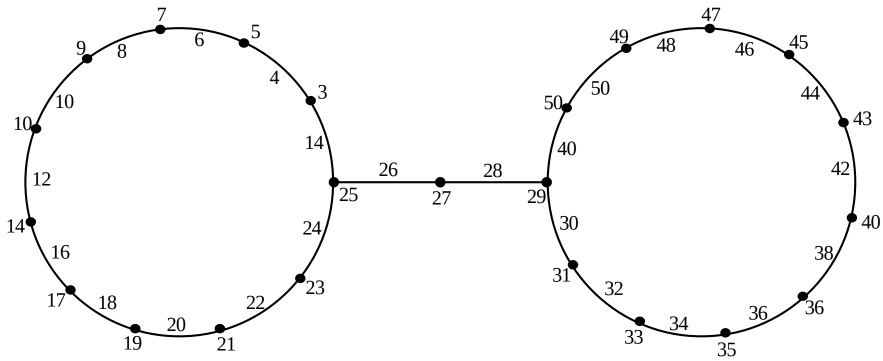

Fig. 2.6: 2-*EML* of $D_{3,11}$ @ C_{11}

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Graceful Labeling of Generalized Tree of Hanging Stars in Arithmetic Progression

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Abstract: *In this paper, it is shown graceful a labeling of generalized tree obtained from a family of n stars having number of branches of those stars form an arithmetic progression with common difference j and one of the branches of each of those stars merged with one different point of a common path on n vertices successively in increasing order.*

Key words: *arithmetic progression, growing stars, supporting points, hanging points, free leaves, general difference.*

Mathematical classification 2010: 05C78

1. Introduction

A simple undirected graph G = (V (G), E (G)) with p vertices and q edges. A function f is called graph labeling of graph G if f: V→ {0, 1, 2 … q} is injective and the induced function f*: E→ {1, 2, 3,…,q} defined as f*(e = uv) = $|f(u)-f(v)|$ is bijective. All edge values are unique and distinct.

Gallian, [4] gives extensive survey on graceful labeling. Huang, Kotzig, and Rosa [1] gives a new class of graceful trees, Sethuraman and Jesintha [2] shows a new class of graceful rooted trees, they showed generating new graceful trees[3] and Michelle Edwards and Lea Howard, given a survey of graceful trees[5].

Let P_n be basic path of $T^{(j)}_{(A(n),a)}$ tree. Let $s_1, s_2,\ldots, s_n$ be such vertices, which are termed as supporting vertices of $T^{(j)}_{(A(n),a)}$ tree. In $T^{(j)}_{(A(n),a)}$ at each s_i, a star S_i with i branches having centre c_i with one of the branch vertex of S_i merged with s_i. Here $\{|S_1|, |S_2|,\ldots,|S_n|\}$ forms an arithmetic progression with common difference j and hence it has been denoted as $T^{(j)}_{(A(n),a)}$ tree, where $|E(S_1)| = a$.

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2. Main results

Let the support points of the hanging stars $S_1, S_2, \ldots, S_n$ be $s_1, s_2, s_3, \ldots, s_n$ respectively and denote the free leaves of each of the stars S_i by $f_1^{(i)}, f_2^{(i)}, \ldots, f_{i-1}^{(i)}$ for i = 1, 2, …, n.

Let $c_1, c_2, \ldots, c_n$ be the central vertices of the stars $S_1, S_2, S_3, \ldots, S_n$ respectively.

A tree with growing n hanging stars as branches whose cardinality are in generalized arithmetic progression with common difference j is denoted by $T^{(j)}_{(A(n),a)}$, where 'a' is number of branches in star S_1 and 'j' is common difference between number of branches of any two consecutive stars.

Stars of tree $T^{(j)}_{(A(n),a)}$ can be derived by the relation $|V(S_i)| = a + (i-1)j+1$, where i = 1, 2, 3,…, n.

It can be verified that the number of vertices of $T^{(j)}_{(A(n),a)}$ can be recursively defined by the relation

$$\left|V(T^{(j)}_{(A(n),a)})\right| = \left|V(T^{(j)}_{(A(n-1),a)})\right| + (1+j).$$

Also the edges of $T^{(j)}_{(A(n),a)}$ can be defined by the relation

$$\left|E(T^{(j)}_{(A(n),a)})\right| = \left|E(T^{(j)}_{(A(n-1),a)})\right| + (1+j).$$

Because of the above relation, we define the relation between two successive trees $T^{(j)}_{(A(n-1),a)}$ and $T^{(j)}_{(A(n),a)}$ as

$$\left|T^{(j)}_{(A(n),a)}\right| \Theta \left|T^{(j)}_{(A(n-1),a)}\right| = 1+j,$$

where Θ denote the difference between the number of vertices (edges) of $T^{(j)}_{(A(n-1),a)}$ and $T^{(j)}_{(A(n),a)}$.

Let us assume that $|E(S_1)| = a = q_1$, $|E(S_2)| = a + j = q_2$, $|E(S_3)| = a + 2j = q_3, \ldots$, and $|E(S_n)| = a + (n-1)j = q_n$.

For an example a general tree $T^{(1)}_{(A(n),3)}$ is as shown in figure 1.

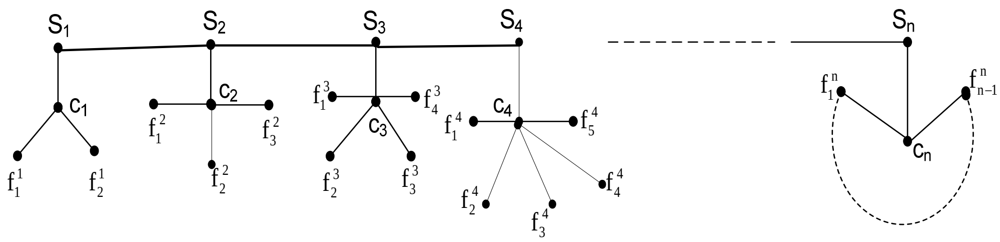


Figure 1

Total number of edges in $T^{(j)}_{(A(n),a)}$ is $q = \frac{n(2a+(n-1)j)}{2} + (n-1)$, $n \geq 1$,

where (n-1) denotes the number of edges in the base path from which the stars are hanging.

We also denote the labeling of node v in the tree as l (v). Here for the tree $T^{(j)}_{(A(n),a)}$ we assign the labeling as follows.

R (1): $l(s_1) = 0$; $l(c_1) = q$; $l(c_2) = a$, $l(s_2) = q-a$.

R (2): $l(s_{2m+1}) = l(s_{2m-1}) + (j(2m-1) + 1 + a)$, $m \geq 1$.

R (3): $l(s_{2m+2}) = l(s_{2m}) - (2mj + 1 + a)$, $m \geq 1$.

R (4): $l(c_{2m+1}) = l(c_{2m-1}) - (j(2m-1) + 1 + a)$, $m \geq 1$.

R (5): $l(c_{2m+2}) = l(c_{2m}) + (2mj + 1 + a)$, $m \geq 1$.

Let the free leaves of growing m^{th} star of $T^{(j)}_{(A(n),a)}$ at s_m be f_1^m, f_2^m… f_k^m, where $k = a + (m-1)j - 1$.

Let the free leaves of S_1 are labeled with values 1 to q_1-1.

Then for $m \geq 1$

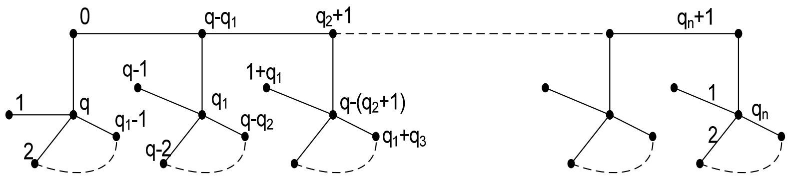
The labeling of free leaves of odd stars of S_{2m+1} based on its supporting vertex s_{2m+1} as follows.

R(6)a: labeling of (a+ 2mj-1) free leaves of S_{2m+1} are given by the integers starting from $l(c_{2m}) + 1$ to $(l(c_{2m}) + a + 2mj)$ except the value of $l(s_{2m+1})$.

The labeling of free leaves of even stars S_{2m} based on its supporting vertex s_{2m} as follows.

R(6)b: labeling of (a+(2m-1)j-1)free leaves of S_{2m} are given by the integers starting from $(l(c_{2m-1}) - (a + (2m-1)j))$ to $(l(c_{2m-1}) - 1)$ except the value of $l(s_{2m})$.

The edge assignment follows.



The above labeling of edges induces a bijective mapping I_E and I_V as follows.

I_E: E ($T^{(j)}_{(A(n),a)}$) $\to$ $\{1, 2, 3,\ldots, \frac{n(|V(S_1)|+|V(S_n)|)}{2} + n - 1\}$

I_V: V ($T^{(j)}_{(A(n),a)}$) $\to$ $\{0, 1, 2,\ldots, \frac{n(|V(S_1)|+|V(S_n)|)}{2} + n - 1\}$

and it can be verified easily that it is a graceful labeling for the given tree $T^{(j)}_{(A(n),a)}$ from the following vertex and edge assignments tables.

The vertex assignment table:

Labeling of T_n	Labeling of vertices
s_1	0
c_1	q
Remaining free leaves of S_1	1 to a-1
s_2	q-a
c_2	a
Remaining free leaves of S_2	{q-1 to (q-q_2) except (q-a)}
s_3	1+q_2
c_3	q-(1+q_2)
Remaining free leaves of S_3	{(a +1) to (a + q_3)except(1+q_2))}
s_4	q-(a+q_3+1)
c_4	a+ q_3 + 1
Remaining free leaves of S_4	{(q-q_2-2) to (q-(q_2+q_4+1)except (q-(a+q_3+1)}
s_{2m}	q-l(c_{2m})
c_{2m}	a+q_3+q_5+…+q_{2m-1}+m-1
Remaining free leaves of S_{2m}	The relation R (6b) assigns the values.
s_{2m+1}	q_2+q_4+…+q_{2m}+m
c_{2m+1}	q-l(s_{2m+1})
Remaining free leaves of S_{2m+1}	The relation R (6a) assigns the values.
s_{2n}	q- l(c_{2n})
c_{2n}	l(c_{2n-2}) + 1+a+(2n-2)j
Remaining free leaves of S_{2n}	The relation R (6b) assigns the values.
s_{2n+1}	l(s_{2n-1}) +1+a+ (2n-1)j
c_{2n+1}	q- l(s_{2n+1})
Remaining free leaves of S_{2n+1}	The relation R (6a) assigns the values.

Table 1

The edge assignment table:

Labeling of T_n	Labeling of edge values
edge s_1c_1	q
free leaves of S_1	q-1 to q-(a-1)
edge s_1s_2	q-a
edge s_2c_2	q-2a
free leaves of S_2	q-(a+1) to q-(a+q_2) except (q-2a)
edge s_2s_3	q-(a+q_2+1)
edge s_3c_3	q-(2q_2+2)
free leaves of S_3	q-(a+q_2+2) to q-(a+q_2+q_3+1) except q-2(1+q_2)
edge s_3s_4	q-(a+q_2+q_3+2)
edge s_4c_4	q-2(a+q_3+1)
free leaves of S_4	q-(a+q_2+q_3+3) to q-(a+q_2+q_3+q_4+2) except q-2(a+q_3+1)
edge $s_{2m-1}s_{2m}$	q-(a+q_2+…+q_{2m-1}+2m-2)
edge $s_{2m}c_{2m}$	q-2(a+q_2+…+q_{2m-1}+m-1)
free leaves of S_{2m}	q-(a+q_2+…+q_{2m-1}+2m-1) to q-(a+q_2+…+q_{2m}+2m-2)except edge l($s_{2m}c_{2m}$)
edge $s_{2m}s_{2m+1}$	q-(a+q_2+…+q_{2m}+2m-1)
edge $s_{2m+1}c_{2m+1}$	q-2(a+q_2+…+q_{2m}+m)
free leaves of S_{2m+1}	q-(a+q_2+…+q_{2m}+2m) to q-(a+q_2+…+q_{2m+1}+2m-1) except edge l($s_{2m+1}c_{2m+1}$)
edge $s_{n-1}s_n$	1+|E(S_n)|
edge s_nc_n	|l(s_n)-l(c_n)|
free leaves of S_n	|E (S_n)| to 1 except edge l(s_nc_n)

Table 2

We observe that the labeling of S_i's in which s_{2m+1}, m ≥ 1 are increasing order and s_{2m}, m ≥ 2 are decreasing order in relation with q. l(s_i) + l(c_i) = q. for any i, and hence it can be observed that the reverse property of l(s_i)'s is satisfied for l(c_i)'s.

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Graceful Labeling of Generalized Tree with Hanging Stars in Geometric Progression

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Abstract: *In this paper, it is shown graceful a labeling of generalized tree obtained from a family of n stars having number of branches of those stars, form a geometric progression with common ratio r and one of the branches of each of those stars, merged with one different point of a common path on n vertices successively in increasing order.*

Key words: *geometric progression, growing stars, supporting points, hanging points, free leaves, general ratio.*

1. Introduction

A simple undirected graph G = (V (G), E (G)) with p vertices and q edges. A function f is called graph labeling of graph G if f: V→ {0, 1, 2 … q} is injective and the induced function f*: E→ {1, 2, 3,…, q} defined as f*(e = uv) = $|f(u)-f(v)|$ is bijective. All edge values are unique and distinct.

Gallian, [4] gives extensive survey on graceful labeling. Huang, Kotzig, and Rosa [1] gives a new class of graceful trees, Sethuraman and Jesintha [2] shows a new class of graceful rooted trees, they showed generating new graceful trees [3] and Michelle Edwards and lea Howard, given a survey of graceful trees[6]. In the earlier paper of arithmetic progression by us [5] motivated to find graceful labeling for trees with hanging stars which are in geometric progression.

Let P_n be basic of $T^{(r)}_{(G(n),a)}$ tree. Let $s_1, s_2,\dots, s_n$ be such vertices, which are, term it as supporting vertices of $T^{(r)}_{(G(n),a)}$ tree. In $T^{(r)}_{(G(n),a)}$ at each s_i, a star S_i with i branches having centre c_i with one of the branch vertex of S_i merged with s_i. Here $|S_1|, |S_2|,\dots, |S_n|$ form geometric progression with common ratio r and hence it has been denoted as $T^{(r)}_{(G(n),a)}$ tree, where $|E (S_1)| = a$.

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2. Main results

Let the support points of the hanging stars $S_1, S_2, \ldots, S_n$ be $s_1, s_2, s_3, \ldots, s_n$ respectively and denote the free leaves of each of the stars S_i by $f_1^{(i)}, f_2^{(i)}, \ldots, f_{i-1}^{(i)}$ for $i = 1, 2, \ldots, n$.

Let $c_1, c_2, \ldots, c_n$ be the central vertices of the stars $S_1, S_2, S_3, \ldots, S_n$ respectively.

A tree with growing n hanging stars as branches whose cardinality are in geometric progression with common ratio 'r' is denoted by $T^{(r)}_{(G(n),a)}$, where 'a' is number of branches in star S_1 and 'r' is common ratio any two consecutive stars.

Stars of tree $T^{(r)}_{(G(n),a)}$ can be derived by the relation $|V(S_n)| = a\, r^{n-1} + 1$, where n= 1, 2…

It can be verified that the number of vertices of $T^{(r)}_{(G(n),a)}$ can be recursively defined by the relation

$$\left|V(T^{(r)}_{(G(n),a)})\right| = \left|V(T^{(r)}_{(G(n-1),a)})\right| + (1 + r^n) \ .$$

Also the edges of $T^{(r)}_{(G(n),a)}$ can be defined by the relation $\left|E(T^{(r)}_{(G(n),a)})\right| = \left|E(T^{(r)}_{(G(n-1),a)})\right| + (1 + r^n)$.

Because of the above relation, we define the relation between two successive trees $T^{(r)}_{(G(n),a)}$ and $T^{(r)}_{(G(n),a)}$ as $\left|T^{(r)}_{(G(n),a)}\right| \Theta \left|T^{(r)}_{(G(n-1),a)}\right| = 1 + r^n$,

where Θ denote the difference between the number of vertices (edges) of $T^{(r)}_{(G(n),a)}$ and $T^{(r)}_{(G(n),a)}$.

let us assume that $|E(S_1)| = a = q_1$, $|E(S_2)| = a\,r = q_2$, $|E(S_3)| = ar^2 = q_3, \ldots,$ and $|E(Sn)| = ar^{n-1} = q_n$.

For example a general tree $T^{(2)}_{(G(n),2)}$ drawn in Figure 1.

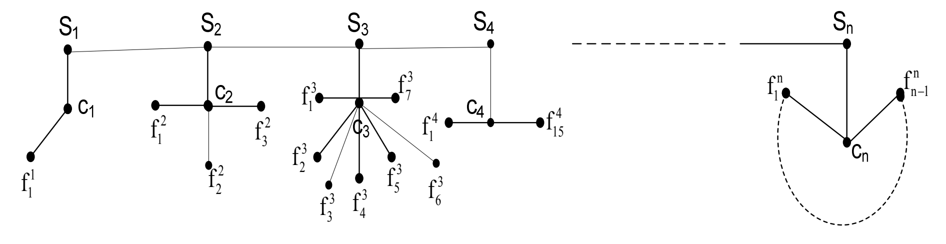


Figure 1

Total number of edges q = $\frac{a(1-r^n)}{1-r}+(n-1)$.

where (n-1) denotes the number of edges in the base path from which the stars are hanging.

We also denote the labeling of node v in the tree as l (v). Here for the tree $T^{(r)}_{(G(n),a)}$, we assign the labeling as follows.

R (1): $l(s_1) = 0$; $l(c_1) = q$; $l(c_2) = a$, $l(s_2) = q-a$.

R (2): $l(s_{2m+1}) = l(s_{2m-1}) + (a\,r^{2m-1} + 1)$, $m \ge 1$.

R (3): $l(s_{2m+2}) = l(s_{2m}) - (a\,r^{2m} + 1)$, $m \ge 1$.

R (4): $l(c_{2m+1}) = l(c_{2m-1}) - (a\,r^{2m-1} + 1)$, $m \ge 1$.

R (5): $l(c_{2m+2}) = l(c_{2m}) + (a\,r^{2m} + 1)$, $m \ge 1$.

Let the free leaves of growing m^{th} star of $T^{(r)}_{(G(n),a)}$ at s_m be $f_1^m, f_2^m \ldots f_k^m$ where $k = ar^{m-1}-1$.

Let the free leaves of S_1 are labeled with values 1 to q_1-1.

Then for $m \ge 1$

The labeling of free leaves of odd stars of S_{2m+1} based on its supporting vertex s_{2m+1} as follows.

R(6)a: labeling of $(ar^{2m}-1)$ free leaves of S_{2m+1} are given by the integers starting from $l(c_{2m}) + 1$ to $l(c_{2m}) + ar^{2m}$ except the value of $l(s_{2m+1})$.

The labeling of free leaves of even stars S_{2m} based on its supporting vertex s_{2m} as follows.

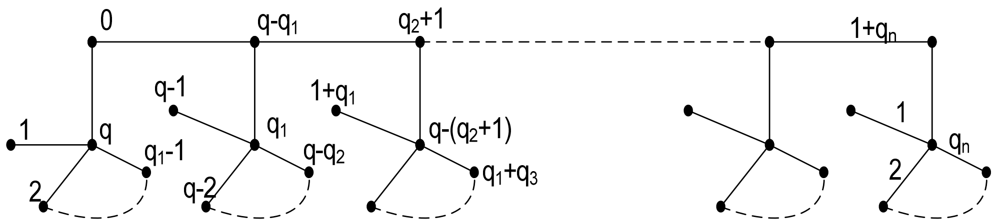
R (6) b: labeling of $(ar^{2m-1}-1)$ free leaves of S_{2m} are given by the integers starting from $l(c_{2m-1}) - ar^{2m-1}$ to $l(c_{2m-1}) - 1$ except the value of $l(s_{2m})$.

The above labeling of vertices (edges) induces a bijective mapping I_E and I_V as follows.

I_E: E ($T^{(r)}_{(G(n),a)}$) $\longrightarrow \{1, 2, 3, \ldots, \frac{a(1-r^n)}{1-r}+(n-1)\}$

I_V: V ($T^{(r)}_{(G(n),a)}$) $\longrightarrow \{0, 1, 2, \ldots, \frac{a(1-r^n)}{1-r}+(n-1)\}$

The edge assignment as follows.



this could be verified easily that it is a graceful labeling for the given tree $T^{(r)}_{(G(n),a)}$ from the following assignment tables.

The vertex assignment table:

Labeling of T_n	Labeling of vertices
s_1	0
c_1	q
Remaining free leaves of S_1	1 to a-1
s_2	q-a
c_2	a
Remaining free leaves of S_2	{q-1 to (q-q_2) except (q-a)}
s_3	1+q_2
c_3	q-(1+q_2)
Remaining free leaves of S_3	{(a +1) to (a + q_3) except(1+q_2))}
s_4	q-(a+q_3+1)
c_4	a+ q_3 + 1
Remaining free leaves of S_4	{(q-q_2-2) to (q-(q_2+q_4+1) except (q-(a+q_3+1)}
s_{2m}	q-l(c_{2m})
c_{2m}	a+q_3+q_5+…+q_{2m-1}+m-1
Remaining free leaves of S_{2m}	The relation R (6b) assigns the values.
s_{2m+1}	q_2+q_4+…+q_{2m}+m
c_{2m+1}	q-l(s_{2m+1})
Remaining free leaves of S_{2m+1}	The relation R (6a) assigns the values.

s_{2n}	q- $l(c_{2n})$
c_{2n}	$l(c_{2n}) = l(c_{2n-2}) + a\, r^{2n-2} + 1$
Remaining free leaves of S_{2n}	The relation R (6b) assigns the values.
s_{2n+1}	$l(s_{2n+1}) = l(s_{2n-1}) + ar^{2n-1} + 1)$
c_{2n+1}	q- $l(s_{2n+1})$
Remaining free leaves of S_{2n+1}	The relation R (6a) assigns the values.

Table 1

The edge assignment follows.

Labeling of T_n	Labeling of edge values
edge s_1c_1	q
free leaves of S_1	q-1 to q-(a-1)
edge s_1s_2	q-a
edge s_2c_2	q-2a
free leaves of S_2	$q-(a+1)$ to $q-(a+q_2)$ except $(q-2a)$
edge s_2s_3	$q-(a+q_2+1)$
edge s_3c_3	$q-(2q_2+2)$
free leaves of S_3	$q-(a+q_2+2)$ to $q-(a+q_2+q_3+1)$ except $q-2(1+q_2)$
edge s_3s_4	$q-(a+q_2+q_3+2)$
edge s_4c_4	$q-2(a+q_3+1)$
free leaves of S_4	$q-(a+q_2+q_3+3)$ to $q-(a+q_2+q_3+q_4+2)$ except $q-2(a+q_3+1)$
edge $s_{2m-1}s_{2m}$	$q-(a+q_2+\ldots+q_{2m-1}+2m-2)$
edge $s_{2m}c_{2m}$	$q-2(a+q_2+\ldots+q_{2m-1}+m-1)$
free leaves of S_{2m}	$q-(a+q_2+\ldots+q_{2m-1}+2m-1)$ to $q-(a+q_2+\ldots+q_{2m}+2m-2)$ except edge $l(s_{2m}c_{2m})$
edge $s_{2m}s_{2m+1}$	$q-(a+q_2+\ldots+q_{2m}+2m-1)$
edge $s_{2m+1}c_{2m+1}$	$q-2(a+q_2+\ldots+q_{2m}+m)$
free leaves of S_{2m+1}	$q-(a+q_2+\ldots+q_{2m}+2m)$ to $q-(a+q_2+\ldots+q_{2m+1}+2m-1)$ except edge $l(s_{2m+1}c_{2m+1})$

edge $s_{n-1}s_n$	1+ | E (S_n)|
edge s_nc_n	|l(s_n)-l(c_n)|
free leaves of S_n	| E (S_n)| to 1 except edge l(s_nc_n)

Table 2

We observe that the labeling of S_i's in which s_{2m+1}, m ≥ 1 are increasing order and s_{2m}, m ≥ 2 are decreasing order in relation with q. l(s_i) + l(c_i) = q. for any i, and hence it can be observed that the reverse property of l(s_i)'s is satisfied for l(c_i)'s.

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On Super Duplicate Graphs

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Abstract: *For a graph G, let V(G) and E(G) denote its vertex set and edge set respectively. Let V′(G) = {v′ : v∈V(G)} be a copy of V(G). The Super duplicate graph D*(G) of G is the graph whose vertex set is V(G)∪V′(G) and edge set is* $E(\overline{G}) \cup \{u'v, uv' : uv \in V(G)\}$*, where* $\overline{G}$ *is the complement of G. In this paper, some basic properties of D*(G) are studied. Also a criterion for D*(G) to be Eulerian and a sufficient condition for Hamiltonicity are obtained. Finally, the parameters girth, connectivity, covering number, independence number, chromatic number, domination number and neighborhood number are determined for super duplicate graphs.*

1. Introduction

Graphs discussed in this paper are undirected and simple graphs. For a graph G, let V(G) and E(G) denote its vertex set and edge set respectively. *Eccentricity* of a vertex $u \in V(G)$ is defined as $e_G(u) = \max\{d_G(u, v): v \in V(G)\}$, where $d_G(u, v)$ is the distance between u and v in G. If there is no confusion, then we simply denote the eccentricity of vertex v in G as e(v) and d(u, v) to denote the distance between two vertices u, v in G respectively. The minimum and maximum eccentricities are the *radius* and *diameter* of G, denoted r(G) and diam(G) respectively. When diam(G) = r(G), G is called a *self-centered* graph with radius r, equivalently G is r-self-centered.

The concept of domination in graphs was introduced by Ore [4]. A set $D \subseteq V(G)$ is said to be a dominating set of G, if every vertex in $V(G)-D$ is adjacent to some vertex in D. D is said to be a minimal dominating set if $D-\{u\}$ is not a dominating set, for any $u \in D$. The domination number $\gamma(G)$ of G is the minimum cardinality of a dominating set. D is a global dominating set, if it is a dominating set of both G and its complement $\overline{G}$. The global domination number γ_g of G is the minimum cardinality of a global dominating set [7]. A *total dominating set* D of G is a dominating set such that the induced sub graph <D> has no isolated vertices. The *total domination number* $\gamma_t(G)$ of G is the minimum

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cardinality of a total dominating set. This concept was introduced in Cockayne *et al* [1]. A γ-set is a minimum dominating set. Similarly, a γ_g-set, γ_t-set are defined.

For $v \in V(G)$, the neighborhood $N(v)$ of v is the set of all vertices adjacent to v in G. $N[v] = N(v) \cup \{v\}$ is called the closed neighborhood of v. A subset S of V(G) is a *neighborhood set* (n-set) of G, if $G = \cup_{v \in S} \langle N[v] \rangle$, where $\langle N[v] \rangle$ is the sub graph of G induced by N[v]. The *neighborhood number* $n_0(G)$ of G is the minimum cardinality of an n-set of G [8].

For a graph G, let $V'(G) = \{v' : v \in V(G)\}$ be a copy of V(G). Then the Duplicate graph D(G) of G is the graph whose vertex set is $V(G) \cup V'(G)$ and edge set is $\{u'v$ and $uv' : uv \in E(G)\}$. This graph was first studied by Sampathkumar [6] and was further developed by Patil et al [5]. The *Super duplicate graph* $D^*(G)$ of G is the graph whose vertex set is $V(G) \cup V'(G)$ and edge set is $E(\overline{G}) \cup \{u'v, uv' : uv \in V(G)\}$, where $\overline{G}$ is the complement of G.

The concept of super duplicate graph of a given graph defines Boolean function of a graph based on the adjacency of the vertices of the given graph. The important application of facility location on networks is based on various types of graphical centrality, all of which are defined using distance. There has been rapid growth of research in the study of domination parameters of graphs, it is used in communication network, coding theory and in network surveillance by Radar stations; it finds application in Projective Geometry and in 'covering' or 'location problems'.

In this paper, some basic properties of $D^*(G)$ are studied. Also a criterion for $D^*(G)$ to be Eulerian and a sufficient condition for Hamiltonicity are obtained. Finally, the parameters girth, connectivity, covering number, independence number, chromatic number, domination number and neighborhood number are determined for super duplicate graphs. The definitions and details not furnished in this paper may found in [2].

2. Prior Results

In this section, we list some results with indicated references, which will be used in the subsequent main results.

Theorem 2.1[2]: For any nontrivial connected graph G, $\alpha_0 + \beta_0 = p = \alpha_1 + \beta_1$, where p is the number of vertices in G.

Theorem 2.2[5]: If G is either a complete graph $K_n(n \geq 3)$ or G contains an odd Hamiltonian cycle, then the Duplicate graph D(G) is Hamiltonian.

Proposition 2.3[8]: For a graph G of order p, neighborhood number n_0 of G is 1 if and only if G has a vertex of degree p-1.

Theorem 2.4 [3]: Let G be geodetic. Then $\overline{G}$ is also geodetic if and only if G is one of the following.

1. G is a cycle C_5 of length 5.
2. G is a path P_4 of length 3.
3. G is isomorphic to the Bull graph.

3. Main Results

The following elementary properties of a super duplicate graph are immediate. Let G be a (p, q) graph.

(a). Duplicate graph D(G) is spanning sub graph of D*(G). D*(G) is a (2p, ((p(p-1))/2) + q) graph.

(b). V(D*(G)) can be partitioned into two sets V and V' such that the sub graph of D*(G) induced by the vertices in V' is totally disconnected and that of D*(G) induced by the vertices in V is the complement of G.

(c). For any vertex $v \in V(G)$, there are two vertices v and v' in D*(G), such that $\deg_{D^*(G)}(v) = p - 1$ and $\deg_{D^*(G)}(v') = \deg_G(v)$. Hence, $\Delta(D^*(G)) = p - 1$ and $\delta(D^*(G)) = \delta(G)$.

(d). D*(G) is biregular with degree sequence k, p - 1 if and only if G is k- regular, where $k \neq p - 1$ and is regular if and only if G is complete.

(e). If G or $\overline{G}$ is non-planar, then D*(G) is also non-planar.

In the following, we find the graphs G for which D*(G) is disconnected.

Theorem 3.1: Let G be a graph with $\delta(G) \geq 1$. Then D*(G) is disconnected if and only if G is a complete bipartite graph.

Proof: Assume $\delta(G) \geq 1$ and D*(G) is disconnected. Since $\delta(G) \geq 1$ and $\overline{G}$ is an induced sub graph of D*(G), D*(G) is connected, if $\overline{G}$ is connected. Let $\overline{G}$ be disconnected and let $G_1, G_2, \ldots, G_n$, $(n \geq 2)$ be the components of $\overline{G}$. If one of them say,

G_1 is not complete, then there exists a vertex $v \in V(G_1)$ such that $v' \in V(D^*(G))$ is adjacent to at least one vertex in each component of $\overline{G}$ and hence in $D^*(G)$. Thus, $D^*(G)$ is connected. This is a contradiction. Hence, the components G_i $(1 \leq i \leq n)$ are complete in $\overline{G}$. Similarly, if $n \geq 3$, then for each $v \in V(G_i)$ $(1 \leq i \leq n)$, the vertex v' in $D^*(G)$ is adjacent to all the vertices in the remaining components. Hence, $D^*(G)$ is connected, which contradicts the fact that $D^*(G)$ is disconnected. Thus, $n = 2$ and G is a complete bipartite graph. Converse follows easily.

Remark 3.1: If $\delta(G) = 0$, then $\delta(D^*(G)) = 0$.

Lemma 3.1: $D^*(G)$ contains triangles if and only if either $\beta_0(G) \geq 3$ or G contains P_3; a path on 3 vertices as an induced sub graph, where $\beta_0(G)$ is the point independence number of G.

Proof: Assume $D^*(G)$ contains triangles. If $\overline{G}$ has triangles, then $\beta_0(G) \geq 3$. Let $\overline{G}$ be triangle-free. Since any two v_i' 's are nonadjacent in $D^*(G)$, any triangle in $D^*(G)$ will contain two vertices in $V(G)$ and one vertex in $V'(G)$. Then G contains P_3; a path on 3 vertices as an induced sub graph. Hence the lemma follows. Converse is obvious.

In the following, the solution for super duplicate graph, which is bipartite is obtained.

Theorem 3.2: For a connected graph G, $D^*(G)$ is bipartite if and only if G is complete.

Proof: Let $D^*(G)$ be bipartite. Then every cycle in $D^*(G)$ is of even length. By Lemma 3.1., if either $\beta o(G) \geq 3$ or G contains P_3 as an induced sub graph, then $D^*(G)$ is not bipartite. Hence, G is complete. Conversely, if G is complete, then $D^*(G)$ is bipartite with bipartition $[V(G), V'(G)]$.

Remark 3.2: By Theorem 3.2., it follows that $D^*(G)$ is regular bipartite if and only if G is complete.

A connected graph G is said to be ***geodetic***, if a unique shortest path joins any two of its vertices. In the following, the geodetic graphs G for which super duplicate graphs $D^*(G)$ are also geodetic are characterized.

Theorem 3.3: For any geodetic graph G with at least three vertices, $D^*(G)$ is geodetic if and only if G is either P_4; a path on four vertices or C_5; a cycle on five vertices.

Proof: Let G be any geodetic graph with at least three vertices. Then $D^*(G)$ is geodetic if $\overline{G}$ is geodetic, since $\overline{G}$ is an induced sub graph of $D^*(G)$. But $\overline{G}$ is geodetic if and only if G is one of the following graphs: P_4, C_5 and the Bull graph by Theorem 2.4. If G is the Bull graph, then $D^*(G)$ is not geodetic. Hence, G is either P_4 or C_5. Converse follows from the construction of $D^*(G)$.

Next, we prove that the girth of $D^*(G)$ is at most 6.

Theorem 3.4: For any connected graph G having at least three vertices, the girth of $D^*(G)$ is 3, 4 or 6.

Proof: By Lemma 3.1., if either $\beta_0(G) \geq 3$ or G contains P_3 as an induced sub graph, then $D^*(G)$ contains triangles and hence girth of $D^*(G)$ is 3. If not, then G is a complete graph. If G has least four vertices, then girth of $D^*(G)$ is 4. If $G \cong K_3$, then $D^*(G)$ is a cycle on six vertices and hence girth of $D^*(G)$ is 6.

Remark 3.3: Let G be a disconnected graph with $\beta_0(G) = 2$ and does not contain P_3 as an induced sub graph. Then girth of $D^*(G)$ is 4, since if G contains $C_3 \cup K_1$ as an induced sub graph or if

$G \cong 2K_2$, then $D^*(G)$ contains C_4 as an induced sub graph.

Theorem 3.5: $D^*(G)$ is not a tree, for any graph G.

Proof: Follows from Theorem 3.1., Theorem 3.4. and Remark 3.3.

In the following, a criterion for $D^*(G)$ being Eulerian is established.

Theorem 3.6: Let G be any (p, q) graph that is not complete bipartite and $\delta(G) \geq 1$. Then $D^*(G)$ is Eulerian if and only if p is odd and each vertex in G is of even degree.

Proof: Suppose $D^*(G)$ is Eulerian. Then the degree of the vertices v_i and v_i' of $D^*(G)$ are even. But $\deg(v_i)$ in $D^*(G)$ is p-1 and $\deg(v_i')$ in $D^*(G)$ is $\deg_G(v_i)$. Hence, p is odd and each vertex in G is of even degree. Conversely, assume that p is odd and each vertex in G is of even degree. Since G is not complete bipartite and $\delta(G) \geq 1$, $D^*(G)$ is connected. Further by the assumption, every vertex of $D^*(G)$ has even degree. Hence, $D^*(G)$ is Eulerian.

Theorem 3.7: $D^*(G)$ is Hamiltonian if G is either a complete graph or G contains an odd Hamiltonian cycle.

Proof: Since the duplicate graph D(G) of G is a spanning sub graph of $D^*(G)$, the theorem follows by Theorem 2.2.

Theorem 3.8: $D^*(K_n - e)$, ($n \geq 4$ is even) has a Hamiltonian path.

Proof: Let $v_1, v_2, .., v_n$ be the vertices of $K_n - e$, where $n \geq 4$ is even, such that v_2 be non-adjacent to v_4 (say). Then $v_2\, v_3'\, v_{4} \ldots v_n\, v_1'\, v_{n-1}\, v_{n-2}' \ldots v_2'\, v_1\, v_n'$ is a Hamiltonian path in $D^*(K_n - e)$, where $n \geq 4$ is even.

Theorem 3.9: Let G be a graph with $\delta(G) \geq 2$ and $r(G) \geq 2$. Then each edge of $D^*(G)$ lies on a triangle iff

(i). For any two vertices u, v in G with $d_G(u, v) = 3$, $N_{\overline{G}}(u) \cap N_{\overline{G}}(v) \neq \phi$.

(ii). For each edge (u, v) in G, $N_{\overline{G}}(u) \cap N_G(v) \neq \Phi$ and $N_G(u) \cap N_{\overline{G}}(v) \neq \phi$.

Proof: Assume each edge of $D^*(G)$ lies on a triangle. $E(D^*(G)) = E(\overline{G}) \cup \{uv'$ and $u'v : uv \in E(G)\}$. Since $\overline{G}$ is an induced sub graph of $D^*(G)$, edges lying on a triangle in $\overline{G}$ also lie on a triangle in $D^*(G)$. Let $e = (u, v)$ be an edge in $\overline{G}$ not lying in any triangle, where $u, v \in V(G)$. Then $d_G(u, v) \leq 3$. If $d_G(u, v) = 2$, then there exists a vertex $w \in V(G)$ adjacent to both u and v and $u\, w'\, v$ is a triangle in $D^*(G)$ and e lies on a triangle in $D^*(G)$. Since $(u, v) \notin E(G)$, $d_G(u, v) = 3$. Also since $V(D^*(G)) = V(G) \cup V'(G)$ and the duplicate graph D(G) is an induced sub graph of $D^*(G)$, no vertex in $V'(G)$ is adjacent to both u and v in $D^*(G)$. Thus, there must exist a vertex in $\overline{G}$ adjacent to both u and v and hence $N_{\overline{G}}(u) \cap N_{\overline{G}}(v) \neq \phi$. Consider the edge $e' = (u, v')$ in $D^*(G)$, where $(u, v) \in V(G)$. By the assumption, e' lies on a triangle in $D^*(G)$. Then there exists a vertex w in $D^*(G)$ adjacent to both u and v'. But $w \notin V'(G)$, since $\langle V'(G) \rangle$ is totally disconnected. Therefore, $w \in V(G)$ and hence $(u, w) \notin E(G)$ and $(v, w) \in E(G)$. Thus, $N_{\overline{G}}(u) \cap N_G(v) \neq \phi$. Similarly, the edge $e'' = (u', v)$ lies on a triangle in $D^*(G)$ implies that $N_G(u) \cap N_{\overline{G}}(v) \neq \phi$. Converse follows easily.

For a graph G, let $\kappa(G)$, $\lambda(G)$ and $\delta(G)$ denote respectively the vertex connectivity, edge connectivity and the minimum degree of G. We will use the theorem of Whitney [2]: For any graph G, $\kappa(G) \leq \lambda(G) \leq \delta(G)$.

Theorem 3.10: If G is a graph that is not complete bipartite, then $\kappa(D^*(G)) \leq \delta(G)$ and $\lambda(D^*(G)) \leq \delta(G)$.

Proof: Since $\delta(D^*(G)) = \delta(G)$, the proposition follows.

The bound $\kappa(D^*(G)) \leq \delta(G)$ is attained, when $r(G) = 1$ or G is a cycle on at least 4 vertices and the bound $\lambda(D^*(G)) \leq \delta(G)$ is attained, for all graphs G with $\delta(G) = 1$ and $G \cong K_4 - e$.

Remark 3.4: Let $\{v_i : 1 \leq i \leq t\}$, $(t \leq p)$ be a vertex cut of $\overline{G}$. If $< \overline{G} - \{v_i : 1 \leq i \leq t\}>$ contains exactly two complete components, then $\{v_i, v_i' : 1 \leq i \leq t\}$ is a vertex cut of $D^*(G)$.

In the following, point covering number α_0, point independence number β_0, line covering number α_1, line independence number β_1 and the chromatic number χ for $D^*(G)$ are determined.

Theorem 3.11: For any graph G with p vertices,
(i). $\alpha_0(D^*(G)) = p = \beta_0(D^*(G))$
(ii). $\alpha_1(D^*(G)) = 2\alpha_1(G)$ and $\beta_1(D^*(G)) = 2\beta_1(G)$.

Proof: Let V be the vertex set of G and V' be the set of new points introduced in the construction of $D^*(G)$. Since V' is an independent set with p points, $\beta_0(D^*(G)) \geq p$. Also each vertex in V is adjacent to at least one vertex in V' and hence any independent set in $D^*(G)$ can have at most p points. Thus, $\beta_0(D^*(G)) = p$. Since $D^*(G)$ has 2p points and $\alpha_0(D^*(G)) + \beta_0(D^*(G)) = 2p$, $\alpha_0(D^*(G)) = p$. It remains to prove that $\beta_1(D^*(G)) = 2\beta_1(G)$. It is to be observed that corresponding to each edge uv of G, there are two independent edges uv' and $u'v$ in $D^*(G)$. Thus, each edge of G gives rise to two independent edges in $D^*(G)$. So $\beta_1(G)$ independent edges of G give $2\beta_1(G)$ independent edges in $D^*(G)$ and this is the maximum number of independent edges in $D^*(G)$. Hence, $\beta_1(D^*(G)) = 2\beta_1(G)$. From the equation $\beta_1(D^*(G)) + \alpha_1(D^*(G)) = 2p = 2\beta_1(G) + 2\alpha_1(G)$, it follows that $\alpha_1(D^*(G)) = 2\alpha_1(G)$.

The ***clique cover number*** of G is the minimum number of complete sub graphs of G, needed to cover the vertices of G and is denoted by $\theta(G)$. For any simple graph G, $\chi(G) = \theta(\overline{G})$.

Theorem 3.12: For any graph G having no isolated vertices, $\chi(D^*(G)) = \theta(G)$ or $\theta(G) + 1$, where $\theta(G)$ is the clique cover number of G.

Proof: Since $\overline{G}$ is an induced sub graph of $D^*(G)$, $\chi(D^*(G)) \geq \chi(\overline{G})$ and hence $\chi(D^*(G)) \geq \theta(G)$, since $\chi(\overline{G}) = \theta(G)$.

Case(i): For each vertex $v \in V(G)$, $N_G(v)$ is k-colorable where $k < \chi(\overline{G})$.

Then color the vertex v' in $D^*(G)$ by a color other than k and thus, $D^*(G)$ is $\chi(\overline{G})$-colorable. Hence, $\chi(D^*(G)) \leq \chi(\overline{G}) = \theta(G)$. Therefore, $\chi(D^*(G)) = \theta(G)$.

Case(ii): There exists a vertex $v \in V(G)$ such that $N_G(v)$ is $\chi(\overline{G})(= \theta(G))$- colorable.

Since the sub graph of $D^*(G)$ induced by $\{v' : v \in V(G)\}$ is totally disconnected, $D^*(G)$ is $(\theta(G) + 1)$-colorable.

From Case (i) and Case(ii), it follows that $\chi(D^*(G)) = \theta(G)$ or $\theta(G) + 1$.

Example 3.1:

1. $\chi(D^*(K_n)) = 2$, if $n \geq 3$.
2. $\chi(D^*(C_n)) = 3$, if $n = 4$; and
   $= \chi(\overline{C_n})$, if $n \geq 5$
3. $\chi(D^*(K_{1,n})) = n + 1$, if $n \geq 2$.

In the following, a necessary and sufficient condition for a global dominating set of G to be a dominating set of $D^*(G)$ is found.

Theorem 3.13: Let G be any graph having no isolated vertices and D be a γ_g-set of G. Then $\gamma(D^*(G) \leq \gamma_g(G)$ if and only if $\delta(\langle D \rangle) \geq 1$.

Proof: Let D be a γ_g-set of G and $\delta(\langle D \rangle) \geq 1$. For any vertex v in G, there are two vertices v, v' in $D^*(G)$. Since $\overline{G}$ is an induced sub graph of $D^*(G)$, D dominates all the vertices in $V(\overline{G}) \cap V(D^*(G))$. By the assumption, for each vertex u' in $V'(G)$ there exists a vertex say, w in D such that u' is adjacent to w. Hence, D is a dominating set of $D^*(G)$. Thus, $\gamma(D^*(G)) \leq \gamma_g(G)$. Conversely, assume a γ_g-set D of G is also a dominating set of $D^*(G)$. If $\delta(\langle D \rangle) = 0$, then there exists a vertex v in <D> such that $\deg_{\langle D \rangle}(v) = 0$ and hence D does not dominate the vertex v'. This is a contradiction to the assumption. Hence, $\delta(\langle D \rangle) \geq 1$.

This bound is attained, if $G \cong C_5$.

Remark 3.5: Theorem 3.13., can be restated as follows. Let D be a γ_g-set of a graph G having no isolated vertices. Then $\gamma(D^*(G)) \leq \gamma_g(G)$ if and only if D is a total dominating set of G.

Remark 3.6: In Theorem 3.13., D must be a γ_g-set of G since otherwise, if D is a dominating set of $\overline{G}$ but not of G, then there exists a vertex v in $\overline{G}$ adjacent to all the vertices in D and hence the vertex v' in $D^*(G)$ is not adjacent to any of the vertices in D.

Remark 3.7: Let D be a γ_g-set of G such that <D> contains isolated vertices. If $D'=$ $\{v' \in V'(G)$: $deg_D(v) = 0\}$, then $D \cup D'$ is a dominating set of $D^*(G)$.

Corollary 3.13.1: Let G be any graph having no isolated vertices and D be a γ_g-set of G. Then $\gamma_g(D^*(G)) \leq \gamma_g(G)$ if and only if $\delta(<D>) \geq 1$.

Corollary 3.13.2: $\gamma(D^*(G)) \leq \gamma_t(G)$ if and only if there exists a γ_t-set D of G such that for each $v \in V(G) - D$ there exists an $u \in D$ such that uv is not an edge in G, where $\gamma_t(G)$ is the total domination number of G.

Corollary 3.13.3: $2 \leq \gamma(D^*(G)) \leq p$.

The lower bound is attained, if $G \cong C_4$, $K_{1,n}$ or K_m, where $n \geq 2$ and $m \geq 3$ and the upper bound is attained, if $G \cong K_2$.

Theorem 3.14: $\gamma(D^*(G)) \leq \delta(G) + 1$, if the neighborhood set of a vertex of minimum degree is a dominating set of G.

Proof: Let $v \in V(G)$ be such that $deg_G(v) = \delta(G)$. If $N_G(v)$ is a dominating set of G, then $N_G[v]$ is a dominating set of $D^*(G)$ and hence $\gamma(D^*(G)) \leq \delta(G) + 1$.

This bound is attained, if $G \cong C_5$.

Theorem 3.15: If diam(G) = 2, then $\gamma(D^*(G)) \leq \delta(G) + 1$.

Proof: If diam(G) = 2, then $\gamma(G) \leq \delta(G)$. Let v be a vertex in G such that $deg_G(v) = \delta(G)$. Then N[v] is a dominating set of $D^*(G)$ and hence $\gamma(D^*(G)) \leq \delta(G) + 1$.

Corollary 3.15.1: If diam(G) = 2, then $\gamma_g(D^*(G)) \leq \delta(G) + 1$.

In the following, a condition for a neighborhood set (n-set) of $\overline{G}$ to be an neighborhood set of $D^*(G)$ is found, when G is triangle-free.

Theorem 3.16: Let G be any graph having no isolated vertices and be triangle-free. Then $n_0(D^*(G)) \leq n_0(\overline{G})$ if and only if there exists an n-set D of $\overline{G}$ with $|D| = n_0(\overline{G})$ such that D is a total dominating set of G.

Proof: Let D be an n-set of $\overline{G}$ with $|D| = n_0(\overline{G})$ such that D is a total dominating set of G. Then $D \subseteq V(D^*(G))$. Since D is an n-set of $\overline{G}$, it is enough to prove that the edges uv' and $u'v$ in $D^*(G)$ belong to $\cup_{w\in D}(E<N[w]>)$.

(i). Since $\overline{G}$ is an induced sub graph of $D^*(G)$, the edges $x'y$, xy' in $D^*(G)$ with $x, y \in D$ and $xy \in E(G)$ belong to $\cup_{w\in D}(E<N[w]>)$.

(ii). Let $x \in D$, $y \in V(G)-D$ such that $xy \in E(G)$. Since $x \in D$ the edge xy' in $D^*(G)$ belongs to $E(<N[x]>)$. Since G is triangle-free and $\delta(<D>) \geq 1$, there exists a vertex $z \in D$ such that $xz \in E(G)$ and $yz \notin E(G)$. Therefore, $x'z$, $yz \in E(D^*(G))$ and hence in $E(<N[z]>)$. Thus, $x'y \in E(<N[z]>)$.

(iii). Let $x, y \in V(G)-D$ such that $xy \in E(G)$. Since D is a dominating set of G, there exists a vertex $z \in D$, such that xz is an edge in G. Since G is triangle-free $yz \notin E(G)$. Therefore, the edges $x'z$, yz in $D^*(G)$ belongs to $E(<N[z]>)$. Hence, $x'y \in E(<N[z]>)$. Similarly, $xy' \in E(<N[z]>)$ for some $z \in D$.

From (i), (ii) and (iii), it follows that each edge in $D^*(G)$ belongs to $\cup_{v\in D}(E<N[v]>)$ and D is n-set for $D^*(G)$. Thus, $n_0(D^*(G)) \leq n_0(\overline{G})$. Conversely, assume any n-set D of $\overline{G}$ is also an n-set of $D^*(G)$. If D is not a dominating set of G, then for any two vertices x, $y \in V(G)-D$, with $xy \in E(G)$, the edges $x'y$ and xy' do not belong to $\cup_{z\in D}(E<N[z]>)$. Similarly, if $\delta(<D>) = 0$, then there exists a vertex z in D with $deg_{<D>}(z) = 0$. But since G has no isolated vertices and D is a dominating set of G, there exists a vertex $y \in V(G)-D$ adjacent to z. Hence, the edges $y'z$ and yz' are not in $\cup_{w\in D}(E<N[w]>)$. Thus, $\delta(<D>_G) \geq 1$.

Corollary 3.16.1: Let G be any graph having no isolated vertices and be triangle-free. If there exists an n-set D of $\overline{G}$ with $|D| = n_0(\overline{G})$ such that D is a point cover for G and $\delta(<D>_G) \geq 1$, then D is also an n-set of $D^*(G)$.

In the following, a condition for an n-set of $\overline{G}$ to be an n-set of $D^*(G)$ is found, when G has triangles.

Theorem 3.17: Let G be any graph having no isolated vertices and contain triangles. Then $n_0(D^*(G)) \leq n_0(\overline{G})$ if and only if there exists an n-set D of $\overline{G}$ with $|D| = n_0(\overline{G})$ satisfying

(i). $\delta(\langle D\rangle_G) \geq 1$; and

(ii). For each edge uv in G, at least one of u and v is in $V(G)-D$, $N_G(u)\cap D \neq N_G(v)\cap D$, where both $N_G(u)\cap D$ and $N_G(v)\cap D$ contain at least one vertex in G.

Proof: Let D be an n-set of $\overline{G}$ with $|D| = n_0(\overline{G})$, satisfying conditions (i) and (ii). Since D is an n-set of $\overline{G}$ it is enough to prove the edges of the form xy', $x'y$ in $D^*(G)$ belong to $\cup_{w\in D}(E\langle N[w]\rangle)$, where $x, y\in V(G)$ and $xy\in E(G)$.

(a). For $x, y\in D$ with $xy\in E(G)$, the edges xy', $x'y$ in $D^*(G)$ belong to $\cup_{v\in D}(E\langle N[v]\rangle)$ since D is an n-set of $\overline{G}$.

(b). By conditions (i) and (ii), corresponding to the edge $xy\in E(G)$ with $x\in D$, $y\in V(G)-D$ or both $x, y\in V(G)-D$, the edges xy', $x'y$ in $D^*(G)$ belong to $\cup_{z\in D}(E\langle N[z]\rangle)$. Thus, each edge in $D^*(G)$ belongs to $\cup_{v\in D}(E\langle N[v]\rangle)$ and $D^*(G) = \cup_{v\in D}(E\langle N[v]\rangle)$. Hence, D is an n-set of $D^*(G)$ and $n_0(D^*(G)) \leq n_0(\overline{G})$. Conversely, assume any n-set D of $\overline{G}$ is also an n-set of $D^*(G)$. If condition (i) or (ii) is not true, then D cannot be an n-set of $D^*(G)$.

Corollary 3.17.1: Let G be any graph and D be a dominating set of $\overline{G}$ such that it is also a point cover for G. Then D is an n-set of $D^*(G)$ if and only if for each edge $uv\in E(G)$ with $u\in D$, $v\in V(G)-D$, $N_G(u)\cap D \neq N_G(v)\cap D$, where both $N_G(u)\cap D$ and $N_G(v)\cap D$ are not empty.

Theorem 3.18: For any graph G, $2 \leq n_0(D^*(G)) \leq p$.

Proof: $n_0(D^*(G)) = 1$ if and only if $D^*(G)$ has a vertex of degree $(2p-1)$. Hence, $n_0(D^*(G)) \geq 2$. Also the set $V(G)$ is an n-set for $D^*(G)$ and hence $n_0(D^*(G)) \leq p$.

The lower bound is attained, if G is a star and the upper bound is attained, if G is a complete graph on at least 2 vertices.

Example 3.2:

1. $n_0(D^*(C_n)) = 3$, if $n = 3$; and
   $= n - 2$, if $n \geq 4$.
2. $n_0(D^*(K_{m,n})) = 2$, if $m, n \geq 2$.
3. If G is the wheel on n vertices, then $n_0(D^*(G)) = n - 2$, if $n \geq 5$.

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Directed Edge–Graceful Labeling of the Graph $T_{t,n,m}$

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Abstract: *Rosa [14] introduced the notion of graceful labelings. The concept of magic, antimagic and conservative labelings have been extended to directed graphs [12]. Bloom and Hsu [3, 4, 5] extended the notion of graceful labeling to directed graph. In 1985, Lo [13] introduced the notion of edge - graceful graphs. We introduced [8] the concept of edge - graceful labelings to directed graphs and further studied in [9,10]. In this paper we investigate directed edge - graceful labeling of $T_{t,n,m}$ graph.*

Keywords: *Directed edge - graceful labeling, Directed edge - graceful graphs.*

AMS (MOS) Subject Classification: 05C78.

1. Introduction

All graphs in this paper are finite and directed. Terms not defined here are used in the sense of Harary [11]. The symbols $V(G)$ and $E(G)$ will denote the vertex set and edge set of a graph G. The cardinality of the vertex set is called the order of G denoted by p. The cardinality of the edge set is called the size of G denoted by q. A graph with p vertices and q edges is called a (p, q) graph.

A graph labeling is an assignment of integers to the vertices or edges or both subject to certain conditions. Labeled graphs serve as useful models for a broad range of applications such as coding theory, X–ray crystallography, radar, astronomy, circuit design, communication network addressing, database management etc. [1, 2]. A good account on graceful labeling problems and other types of graph labeling problems can be found in the dynamic survey of J.A. Gallian [6].

A graph G is called a graceful labeling if f is an injection from the vertices of G to the set $\{0, 1, 2, ..., q\}$ such that, when each edge xy is assigned the label $|f(x) - f(y)|$, the resulting edge labels are distinct.

A graph $G(V, E)$ is said to be edge - graceful if there exists a bijection f from E to $\{1, 2, ..., |E|\}$ such that the induced mapping f^+ from V to $\{0, 1, ..., |V| - 1\}$ given by, $f^+(x) = (\Sigma f(xy)) \bmod(|V|)$ taken overall edges xy incident at x is a bijection.

A necessary condition for a graph G with p vertices and q edges to be edge - graceful is $q(q + 1) \equiv \frac{p(p+1)}{2} (\bmod p)$. Bloom and Hsu [3, 4, 5] extended the notion of graceful labelling to directed graph. The concept of magic, antimagic and conservative

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labelings have been extended to directed graphs [12]. In [8] we extended the concept of edge - graceful labelings to directed graphs and further studied in [9, 10]. In this paper we investigate directed edge - graceful labeling of $T_{t,n,m}$ graph.

A (p, q) graph G is said to be **directed edge - graceful** if there exists an orientation of G and a labeling f of the arcs A of G with $\{1, 2, ..., q\}$ such that induced mapping g on V defined by, $g(v) = \left[f^{+}(v) - f^{-}(v) \right]$ (mod p) is a bijection where, $f^{+}(v)$ = the sum of the labels of all arcs with head v and $f^{-}(v)$ = the sum of the labels of all arcs with v as tail.

A graph G is said to be **directed edge - graceful graph** if it has directed edge–graceful labelings. Here, we investigate directed edge - graceful labeling of some trees and cycle related graphs.

2. Prior Results

Theorem 2.1: [8] The path P_{2n+1} is directed edge - graceful for all $n \geq 1$.

Theorem 2.2: [8] The cycle graph C_{2n+1} is directed edge - graceful for all $n \geq 1$.

Theorem 2.3: [8] The Butterfly graph B_n is directed edge - graceful if n is odd.

Theorem 2.4: [8] The Butterfly graph B_n is directed edge - graceful if n is even and $n \geq 4$.

Theorem 2.5: [8] The snail graph $SN(2n + 1)$ is directed edge - graceful for all $n \geq 1$.

Theorem 2.6: [8] $\left\langle K_{1,n} : K_{1,n} \right\rangle$ is directed edge - graceful if n is even and $n \geq 4$.

Theorem 2.7: [9] The graph $P_3 \cup K_{1,2n+1}$ is directed edge - graceful for all $n \geq 1$.

Theorem 2.8: [9] The graph P_{2m} @ $K_{1,2n+1}$ is directed edge-graceful for all $m \geq 2$ and $n \geq 1$.

Theorem 2.9: [9] The graph P_{2m+1} @ $K_{1,2n}$ is directed edge-graceful for all $m \geq 1$ and $n \geq 1$.

Theorem 2.10: [10] The fan F_{2n} is directed edge - graceful for all $n \geq 2$.

Theorem 2.11: [10] The star graph $K_{1,2n}$ is directed edge - graceful for all $n \geq 1$.

Theorem 2.12: [10] The wheel graph W_{2n} directed edge - graceful for all $n \geq 2$.

Theorem 2.13: [10] The tortoise graph T_{2n+1} is directed edge - graceful for all $n \geq 2$.

Theorem 2.14: [10] The graph nC_3 snake is directed edge - graceful for $n \geq 2$.

3. Main Results

Definition 3.1

$T_{t,n,m}$ is a graph obtained by joining the centers of $K_{1,n}$ and $K_{1,m}$ by a path P_t. It consists of $t + n + m$ vertices and $t + n + m - 1$.

Theorem 3.2

The graph $T_{t,n,m}$ is directed edge - graceful if t, n and m are odd numbers.

Proof

Let $G = T_{t,n,m}$ and $V[T_{t,n,m}] = \{w_1, w_2, ..., w_t, u_1, u_2, \ldots, u_n, v_1, v_2 \ldots, v_m\}$ be the set of vertices. Now we orient the edges of $T_{t,n,m}$ such that the arc set A is given by,

$A = \left\{(w_{2i-1}, w_{2i}),\ 1 \le i \le \frac{t-1}{2}\right\} \cup \left\{(w_{2i+1}, w_{2i}),\ 1 \le i \le \frac{t-1}{2}\right\} \cup \{(w_1, u_j),\ 1 \le j \le n\} \cup \{(w_t, v_k),\ 1 \le k \le m\}$

The edges and their orientation of $T_{t,n,m}$ are as in Fig. 1.

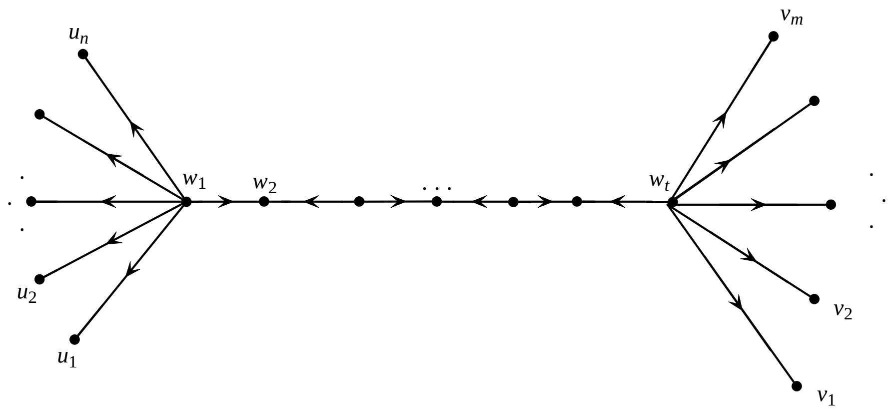

Fig. 1: $T_{t,n,m}$ with orientation

We now label the arcs of A as follows:

$f((w_{2i-1}, w_{2i})) = i,\ 1 \le i \le \frac{t-1}{2}$; $f((w_{2i+1}, w_{2i})) = \frac{t-1}{2} + n + m + i,\ 1 \le i \le \frac{t-1}{2}$

$f((w_1, u_1)) = \frac{t-1}{2} + n + m$; $f((w_t, v_1)) = \frac{t+1}{2}$

$f((w_1, u_{2j})) = \frac{t+m}{2} + j,\ 1 \le j \le \frac{n-1}{2}$; $f((w_1, u_{2j+1})) = \frac{t+m}{2} + n - j,\ 1 \le j \le \frac{n-1}{2}$

$f((w_t, v_{2k})) = \frac{t+1}{2} + k,\ 1 \le k \le \frac{m-1}{2}$; $f((w_t, v_{2k+1})) = \frac{t-1}{2} + n + m - k,\ 1 \le k \le \frac{m-1}{2}$

The values of $f^+(w_i)$, $f^+(u_j)$, $f^+(v_k)$ and $f^-(w_i)$, $f^-(u_j)$ and $f^-(v_k)$ are computed as under.

$f^+(w_1) = 0$; $f^-(w_1) = \frac{-1}{2}[n^2 + n\,(t + m + 1) + m + 1]$

$f^+(w_{2i}) = \frac{t-1}{2} + n + m + 2i,\ 1 \le i \le \frac{t-1}{2}$; $f^-(w_{2i}) = 0$

$f^+(w_{2i+1}) = 0,\ 1 \le i \le \frac{t-3}{2}$; $f^-(w_{2i+1}) = -\left[\frac{t+1}{2} + n + m + 2i\right],\ 1 \le i \le \frac{t-3}{2}$

$f^+(w_t) = 0$; $f^-(w_t) = -\frac{1}{2}[m^2 + m(t + n + 1) + 2t + n - 1]$

$f^+(u_1) = \frac{t-1}{2} + n + m$; $f^-(u_1) = 0$

$f^+(u_{2j}) = \frac{t+m}{2} + j,\ 1 \le j \le \frac{n-1}{2}$; $f^-(u_{2j}) = 0,\ 1 \le j \le \frac{n-1}{2}$

$f^+(u_{2j+1}) = \frac{t+m}{2} + \mathrm{n} - j,\ 1 \le j \le \frac{n-1}{2}$; $f^-(u_{2j+1}) = 0,\ 1 \le j \le \frac{n-1}{2}$

$f^+(v_1) = \frac{t+1}{2}$; $f^-(v_1) = 0$

$f^+(v_{2k}) = \frac{t+1}{2} + k,\ 1 \le k \le \frac{m-1}{2}$; $f^-(v_{2k}) = 0,\ 1 \le k \le \frac{m-1}{2}$

$f^+(v_{2k+1}) = \frac{t-1}{2} + n + m - k,\ 1 \le k \le \frac{m-1}{2}$; $f^-(v_{2k+1}) = 0,\ 1 \le k \le \frac{m-1}{2}$

Then the induced vertex labels are,

$g(u_1) = \frac{t-1}{2} + n + m$; $g(u_{2j}) = \frac{t+m}{2} + j,\ 1 \le j \le \frac{n-1}{2}$

$g(u_{2j+1}) = \frac{t+m}{2} + n - j,\ 1 \le j \le \frac{n-1}{2}$; $g(v_1) = \frac{t+1}{2}$

$g(v_{2k}) = \frac{t+1}{2} + k,\ 1 \le k \le \frac{m-1}{2}$; $g(v_{2k+1}) = \frac{t-1}{2} + n + m - k,\ 1 \le k \le \frac{m-1}{2}$

Case (i): $\frac{t-1}{2}$ **is odd**

$g(w_{2i-1}) = \frac{t+3}{2} - 2i,\ 1 \le i \le \frac{t+1}{4}$; $g(w_{2i}) = \frac{t-1}{2} + n + m + 2i,\ 1 \le i \le \frac{t-3}{4}$

$g\left(w_{\frac{t+1}{2}}\right) = 0$; $g\left(w_{\frac{t-1}{2}+2i}\right) = t + n + m + 1 - 2i,\ 1 \le i \le \frac{t+1}{4}$

$g\left(w_{\frac{t+1}{2}+2i}\right) = 2i,\ 1 \le i \le \frac{t-3}{4}$

Case (ii): $\frac{t-1}{2}$ **is even**

$g(w_{2i-1}) = \frac{t+3}{2} - 2i,\ 1 \le i \le \frac{t-1}{4}$; $g(w_{2i}) = \frac{t-1}{2} + n + m + 2i,\ 1 \le i \le \frac{t-1}{4}$

$g\left(w_{\frac{t+1}{2}}\right) = 0$; $g\left(w_{\frac{t-1}{2}+2i}\right) = 2i - 1,\ 1 \le i \le \frac{t-1}{4}$

$$g\left(w_{\frac{t+1}{2}+2i}\right) = t + n + m - 2i,\ 1 \leq i \leq \frac{t-1}{4}$$

Clearly, $g(V) = \{0, 1, 2, ..., (t + n + m - 1)\} = \{0, 1, 2, ..., p - 1\}$

So, it follows that all the vertex labels are distinct and g is a bijection. Hence, $T_{t,n,m}$ is a directed edge-graceful graph, if t, n and m are odd. The directed edge-graceful labeling of $T_{29,5,7}$ and $T_{31,5,7}$ is given in Fig. (2) and Fig. (3) respectively.

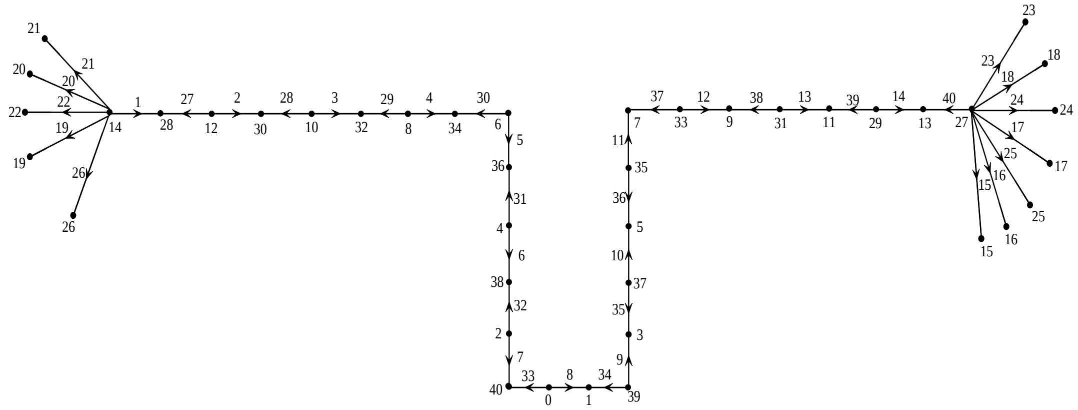

***Fig* (2) : $T_{29,5,7}$ with directed edge-graceful labeling**

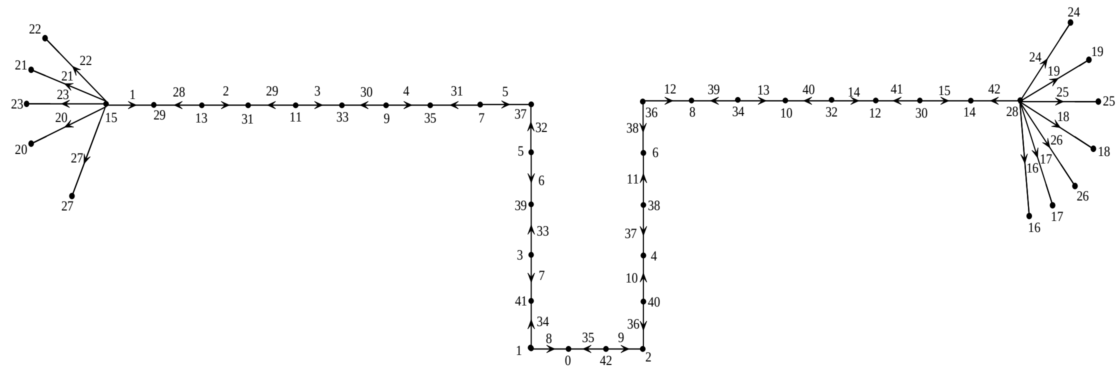

***Fig* (3) : $T_{31,5,7}$ with directed edge-graceful labeling**

Theorem 3.3

The graph $T_{t,n,m}$ is directed – edge graceful if t,n are even and m is odd.

Proof

Let $G = T_{t,n,m}$ and $V[T_{t,n,m}] = \{w_1, w_2, \ldots\ldots w_t, u_1, u_2, \ldots\ldots u_n, v_1, v_2, \ldots, v_m\}$ be the set of vertices. Now we orient the edges of $T_{t,n,m}$ such that the are set A is given by.

$$A = \left\{\left(w_{2i}, w_{2i+1}\right),\ 1 \leq i \leq \frac{t}{2}\right\} \cup \left\{\left(w_{2i}, w_{2i+1}\right),\ 1 \leq i \leq \frac{t}{2} - 1\right\} \cup \{(w_1, u_j), 1 \leq j \leq n\} \cup \{(w_t, v_k), 1 \leq k \leq m\}$$

The edges and their orientation of $T_{t,n,m}$ are as in *Fig* (4).

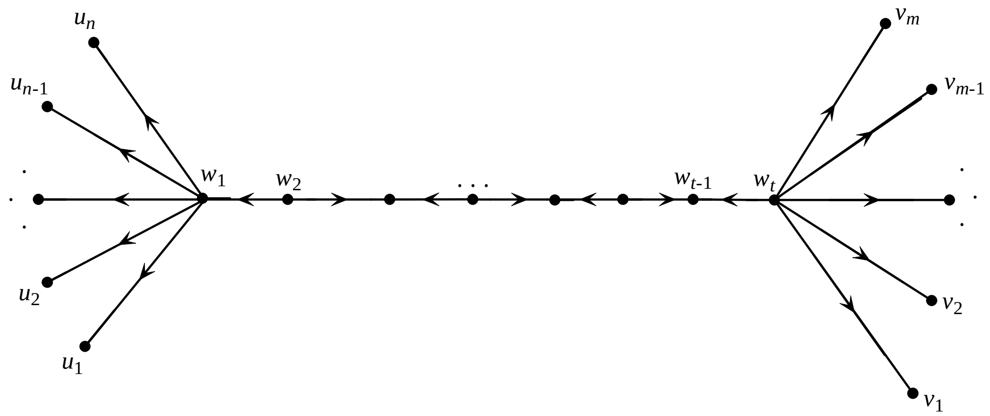

Fig. (4): $T_{t,n,m}$ with orientation

We now label the arcs of A as follows.

$$f\left(\left(w_{2i},\, w_{2i+1}\right)\right) = i,\ 1 \le i \le \frac{t}{2}-1; \qquad f\left(\left(w_{2i},\, w_{2i-1}\right)\right) = \frac{t}{2}+n+m-1+i,\ 1 \le i \le \frac{t}{2}$$

$$f\left(\left(w_1,\, u_{2j-1}\right)\right) = \frac{t}{2}-1+2j,\ 1 \le j \le \frac{n}{2}; \qquad f\left(\left(w_1,\, u_{2j}\right)\right) = \frac{t}{2}+n+m+1-2j,\ 1 \le j \le \frac{n}{2}$$

$$f\left(\left(w_t,\, v_1\right)\right) = \frac{t}{2}\ ; \quad f\left(\left(w_t,\, v_{2k}\right)\right) = \frac{t}{2}+\frac{n}{2}+1+2k,\ 1 \le k \le \frac{m-1}{2}$$

$$f\left(\left(w_t,\, v_{2k+1}\right)\right) = \frac{1}{2}\left(t+n+m+3\right)-2k,\ 1 \le k \le \frac{m-1}{2}$$

The values of $f^{+}(w_i)$, $f^{+}(u_j)$, $f^{+}(v_k)$ and $f^{-}(w_i)$, $f^{-}(u_j)$ and $f^{-}(v_k)$ are computed as under.

$$f^{+}(w_1) = \frac{t}{2}+n+m \quad ; \quad f^{-}(w_1) = -\left[\frac{n}{2}\left(t+n+m\right)\right]$$

$$f^{+}(w_{2i}) = 0,\ 1 \le i \le \frac{t}{2}-1 \quad ; \quad f^{-}(w_{2i}) = -\left[\frac{t}{2}+n+m-1+2i\right],\ 1 \le i \le \frac{t}{2}-1$$

$$f^{+}(w_{2i+1}) = \frac{t}{2}+n+m+2i,\ 1 \le i \le \frac{t}{2}-1 \quad ; \quad f^{-}(w_{2i+1}) = 0,\ 1 \le i \le \frac{t}{2}-1$$

$$f^{+}(w_t) = 0 \quad ; \quad f^{-}(w_t) = -\left[\frac{m^2}{2}+\frac{m}{2}\left(t+n+1\right)+\left(t+\frac{n}{2}-1\right)\right]$$

$$f^{+}(u_{2j-1}) = \frac{t}{2}-1+2j,\ 1 \le j \le \frac{n}{2} \quad ; \quad f^{-}(u_{2j-1}) = 0,\ 1 \le j \le \frac{n}{2}$$

$$f^{+}(u_{2j}) = \frac{t}{2}+n+m+1-2j,\ 1 \le j \le \frac{n}{2} \quad ; \quad f^{-}(u_{2j}) = 0,\ 1 \le j \le \frac{n}{2}$$

$$f^{+}(v_1) = \frac{t}{2} \quad ; \quad f^{-}(v_1) = 0$$

$$f^{+}(v_{2k}) = \frac{t}{2}+\frac{n}{2}+1+2k,\ 1 \le k \le \frac{m-1}{2} \quad ; \quad f^{-}(v_{2k}) = 0,\ 1 \le k \le \frac{m-1}{2}$$

$$f^{+}(v_{2k+1}) = \frac{1}{2}\left(t+n+m-1\right)+2-2k,\ 1 \le k \le \frac{m-1}{2}; \quad f^{-}(v_{2k+1}) = 0,\ 1 \le k \le \frac{m-1}{2}$$

Then the induced vertex labels are,

$g\,(u_{2j-1}) = \frac{t}{2} - 1 + 2j,\ 1 \le j \le \frac{n}{2}$; $g\,(u_{2j}) = \frac{t}{2} + n + m + 1 - 2j,\ 1 \le j \le \frac{n}{2}$

$g\,(v_1) = \frac{t}{2}$; $g\,(v_{2k}) = \frac{t}{2} + \frac{n}{2} + 1 + 2k,\ 1 \le k \le \frac{m-1}{2}$

$g\,(v_{2k+1}) = \frac{1}{2}(t + n + m - 1) + 2 - 2k,\ 1 \le k \le \frac{m-1}{2}$

Case (i): $\frac{t}{2}$ is even

$g\,(w_{2i-1}) = \frac{t}{2} + n + m - 2 + 2i,\ 1 \le i \le \frac{t}{4}$; $g\,(w_{2i}) = \frac{t}{2} + 1 - 2i,\ 1 \le i \le \frac{t}{4}$

$g\left(w_{\frac{t}{2}-1+2i}\right) = 2i - 2,\ 1 \le i \le \frac{t}{4}$; $g\left(w_{\frac{t}{2}+2i}\right) = t + n + m + 1 - 2i,\ 1 \le i \le \frac{t}{4}$

Case (ii) $\frac{t}{2}$ is odd

$g\,(w_{2i-1}) = \frac{t}{2} + n + m - 2 + 2i,\ 1 \le i \le \frac{t+2}{4}$; $g\,(w_{2i}) = \frac{t}{2} + 1 - 2i,\ 1 \le i \le \frac{t+2}{4}$

$g\left(w_{\frac{t}{2}+2i}\right) = 2i - 1,\ 1 \le i \le \frac{t-2}{4}$; $g\left(w_{\frac{t}{2}+1+2i}\right) = t + n + m - 2i,\ 1 \le i \le \frac{t-2}{4}$

Clearly, $g(V) = \{0, 1, \ldots, (t + n + m - 1)\} = \{0,1,\ldots, p - 1\}$

So, it follows that all the vertex labels are distinct and g is a bijection. Hence, $T_{t,n,m}$ is a directed edge - graceful graph if t, n are even and m is odd. The directed edge - graceful labeling of $T_{26,5,5}$ and $T_{28,6,5}$ is given in Fig. (5) and Fig. (6) respectively.

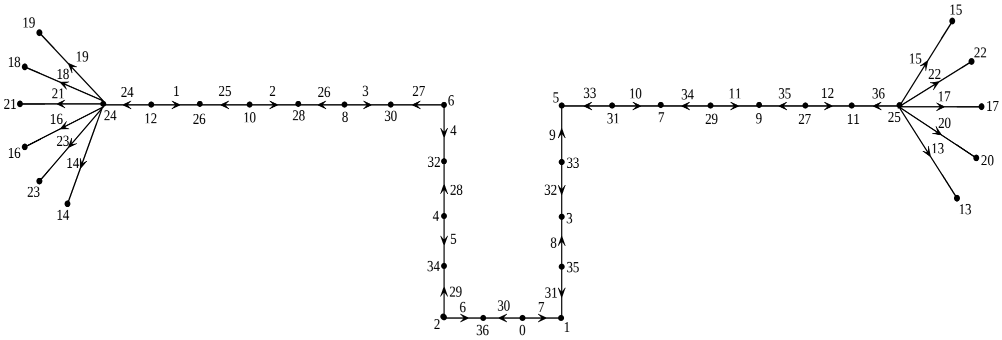

Fig (5) ; $T_{26,6,5}$ with directed edge-graceful labeling

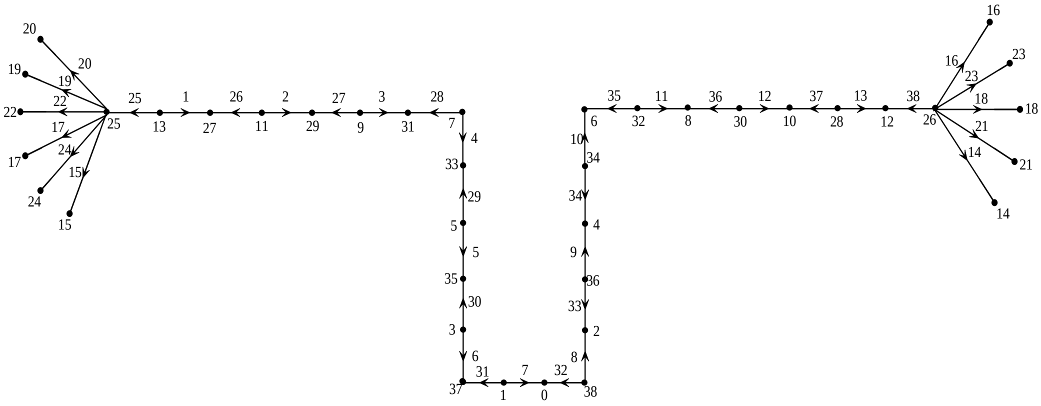

Fig (6) ; $T_{28,6,5}$ with directed edge-graceful labeling

Theorem 3.4

The graph $T_{t,n,m}$ is directed edge-graceful if n and m are even and t is odd.

Proof

Let $G = T_{t,n,m}$ and $V[T_{t,n,m}] = \{w_1, w_2, \ldots, w_t, u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_m\}$ be the set of vertices. Now we orient the edges of $T_{t,n,m}$ such that the are set A is given by,

$A = \left\{(w_{2i-1}, w_{2i}),\ 1 \leq i \leq \dfrac{t-1}{2}\right\} \cup \left\{(w_{2i+1}, w_{2i}),\ 1 \leq i \leq \dfrac{t-1}{2}\right\} \cup \{(w_1, u_j), 1 \leq j \leq n\} \cup \{(w_t, v_k),$

$1 \leq k \leq m\}$

The edges and their orientation of $T_{t,n,m}$ are as in Fig. (7).

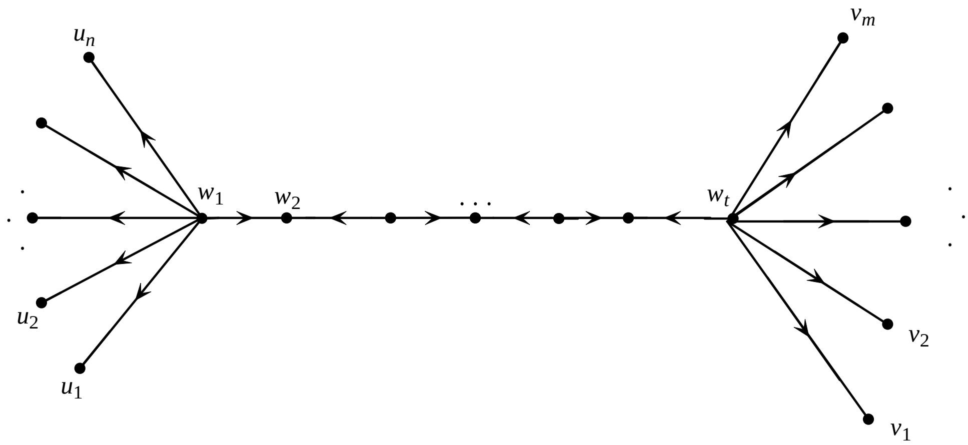

Fig. (7): $T_{t,n,m}$ with orientation

We now label the arcs of A as follows

$f((w_{2i+1}, w_{2i})) = i,\ 1 \leq i \leq \dfrac{t-1}{2}$; $f((w_{2i-1}, w_{2i})) = \dfrac{t-1}{2} + n + m + i,\ 1 \leq i \leq \dfrac{t-1}{2}$

$f((w_1, u_{2j-1})) = \dfrac{t-1}{2} - 1 + 2j,\ 1 \leq j \leq \dfrac{n}{2}$; $f((w_1, u_{2j})) = \dfrac{t-1}{2} + n + m + 2 - 2j,\ 1 \leq j \leq \dfrac{n}{2}$

$f((w_t, v_{2k-1})) = \frac{t-1}{2} + n - 1 + 2k,\ 1 \leq k \leq \frac{m}{2}$; $f((w_t, v_{2k})) = \frac{t-1}{2} + m + 2 - 2k,\ 1 \leq k \leq \frac{m}{2}$

The values of $f^+(w_i)$, $f^+(u_j)$, $f^+(v_k)$ and $f^-(w_i)$, $f^-(u_j)$, $f^-(v_k)$ are computed as under.

$f^+(w_1) = 0$; $f^-(w_1) = -\left[\frac{n^2}{2} + \frac{n}{2}(t+m+2) + \left(\frac{t+2m+1}{2}\right)\right]$

$f^+(w_{2i}) = \frac{t-1}{2} + n + m + 2i,\ 1 \leq i \leq \frac{t-1}{2}$; $f^-(w_{2i}) = 0,\ 1 \leq i \leq \frac{t-1}{2}$

$f^+(w_{2i+1}) = 0,\ 1 \leq i \leq \frac{t-3}{2}$; $f^-(w_{2i+1}) = -\left[\frac{t-1}{2} + n + m + 1\right] + 2i,\ 1 \leq i \leq \frac{t-3}{2}$

$f^+(w_t) = 0$; $f^-(w_t) = -\left[\frac{m}{2}(t+n+m) + \frac{t-1}{2}\right]$

$f^+(u_{2j-1}) = \frac{t-3}{2} + 2j,\ 1 \leq j \leq \frac{n}{2}$; $f^-(u_{2j-1}) = 0,\ 1 \leq j \leq \frac{n}{2}$

$f^+(u_{2j}) = \frac{t+3}{2} + n + m - 2j,\ 1 \leq j \leq \frac{n}{2}$; $f^-(u_{2j}) = 0,\ 1 \leq j \leq \frac{n}{2}$

$f^+(v_{2k-1}) = \frac{t-3}{2} + n + 2k,\ 1 \leq k \leq \frac{m}{2}$; $f^-(v_{2k-1}) = 0,\ 1 \leq k \leq \frac{m}{2}$

$f^+(v_{2k}) = \frac{t+3}{2} + m - 2k,\ 1 \leq k \leq \frac{m}{2}$; $f^-(v_{2k}) = 0,\ 1 \leq k \leq \frac{m}{2}$

Then the induced vertex labels are,

$g(u_{2j-1}) = \frac{t-3}{2} + 2j,\ 1 \leq j \leq \frac{n}{2}$; $g(u_{2j}) = \frac{t-3}{2} + n + m - 2j,\ 1 \leq j \leq \frac{n}{2}$

$g(v_{2k-1}) = \frac{t-3}{2} + n + 2k,\ 1 \leq k \leq \frac{m}{2}$; $g(v_{2k}) = \frac{t+3}{2} + m - 2k,\ 1 \leq k \leq \frac{m}{2}$

Case (i): $\frac{t-1}{2}$ is even

$g(w_{2i-1}) = \frac{t+3}{2} - 2i,\ 1 \leq i \leq \frac{t+3}{4}$; $g(w_{2i}) = \frac{t-1}{2}+n+m+2i,\ 1 \leq i \leq \frac{t-1}{4}$

$g\left[w_{\frac{t-1}{2}+2i}\right] = 2i - 1,\ 1 \leq i \leq \frac{t-1}{4}$; $g\left[w_{\frac{t+1}{2}+2i}\right] = t + n + m - 2i,\ 1 \leq i \leq \frac{t-1}{4}$

Case (ii): $\frac{t-1}{2}$ **is odd**

$g(w_{2i-1}) = \frac{t+3}{2} - 2i,\ 1 \leq i \leq \frac{t+1}{4}$; $g(w_{2i}) = \frac{t-1}{2} + n + m + 2i,\ 1 \leq i \leq \frac{t-3}{4}$

$g\left[w_{\frac{t-3}{2}+2i}\right] = 2i - 2,\ 1 \leq i \leq \frac{t+1}{4}$; $g\left[w_{\frac{t-1}{2}+2i}\right] = t + n + m + 1 - 2i,\ 1 \leq i \leq \frac{t+1}{4}$

Clearly, $g(V) = \{0, 1, \ldots, (t + n + m - 1)\} = \{0, 1, \ldots, p - 1\}$

So, it follows that all the vertex labels are distinct and g is a bijection. Hence, $T_{t,n,m}$ is a directed edge - graceful graph if n and m are even and t is odd. The directed edge - graceful labeling of $T_{27,6,6}$ and $T_{29,6,6}$ is given in Fig. (8) and Fig. (9) respectively.

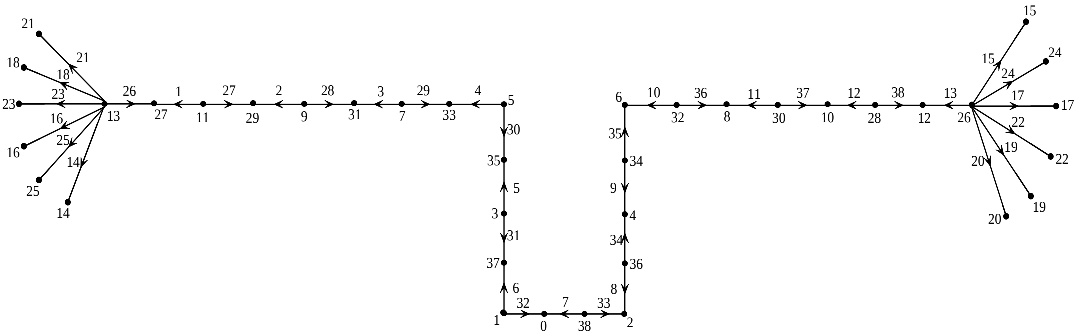

Fig (8) : $T_{27,6,6}$ with directed edge - graceful labeling

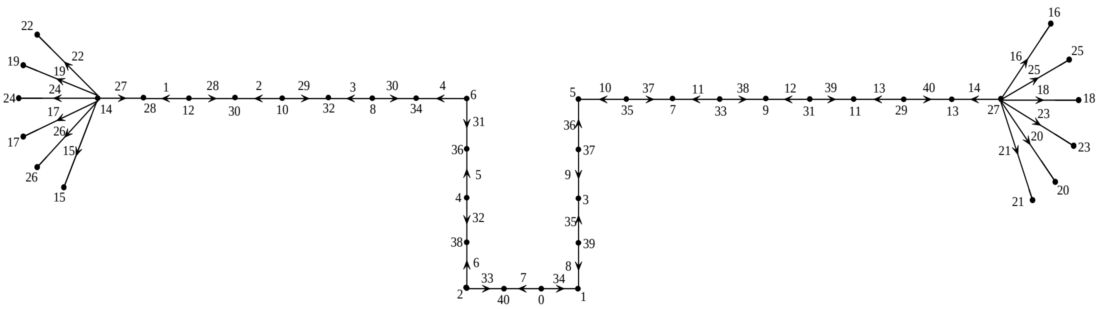

Fig (9) : $T_{29,6,6}$ with directed edge - graceful labeling

Theorem 3.5

The graph $T_{t,n,m}$ is directed edge - graceful if t, m are even and n is odd.

Proof

Let $G = T_{t,n,m}$ and $V[T_{t,n,m}] = \{w_1, w_2, \ldots, w_t, u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_m\}$ be the set of vertices. Now we orient the edges of $T_{t,n,m}$ such that the arc set A is given by,

$$A = \left\{(w_{2i-1}, w_{2i}), 1 \leq i \leq \frac{t}{2}\right\} \cup \left\{(w_{2i+1}, w_{2i}), 1 \leq i \leq \frac{t}{2} - 1\right\} \cup \{(w_1, u_j), 1 \leq j \leq n\} \cup \{(w_t, v_k), 1 \leq k \leq m\}$$

The edges and their orientation of $T_{t,n,m}$ are as in Fig. (10).

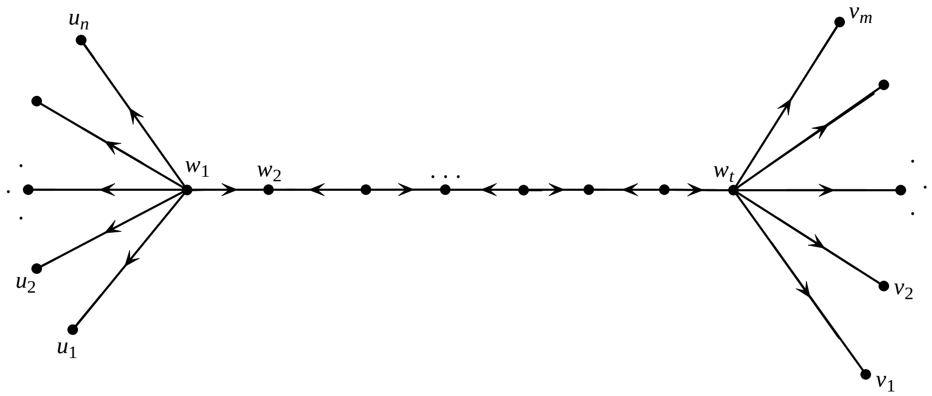

Fig. (10): $T_{t,n,m}$ with orientation

We now label the arcs of A as follows

$f\left(\left(w_{2i-1},\ w_{2i}\right)\right) = i,\ 1 \le i \le \frac{t}{2}$; $f\left(\left(w_{2i+1},\ w_{2i}\right)\right) = \frac{t}{2} + n + m + i,\ 1 \le i \le \frac{t}{2} - 1$

$f\left(\left(w_1,\ u_1\right)\right) = \frac{t}{2} + n + m$; $f\left(\left(w_1,\ u_{2j}\right)\right) = \frac{t}{2} - 1 + 2j,\ 1 \le j \le \frac{n-1}{2}$

$f\left(\left(w_1,\ u_{2j+1}\right)\right) = \frac{t}{2} + n + m + 1 - 2j,\ 1 \le j \le \frac{n-1}{2}$

$f\left(\left(w_t,\ v_{2k-1}\right)\right) = \frac{t}{2} + n - 2 + 2k,\ 1 \le k \le \frac{m}{2}$; $f\left(\left(w_t,\ v_{2k}\right)\right) = \frac{t}{2} + n + 3 - 2k,\ 1 \le k \le \frac{m}{2}$

The values of $f^+(w_i)$, $f^+(u_j)$, $f^+(v_k)$ and $f^-(w_i)$, $f^-(u_j)$ and $f^-(v_k)$ are computed as under.

$f^+(w_1) = 0$; $f^-(w_1) = -\frac{1}{2}\left[n^2 + n(t+m+1) + m + 2\right]$

$f^+(w_{2i}) = \frac{t}{2} + n + m + 2i,\ 1 \le i \le \frac{t}{2} - 1$; $f^-(w_{2i}) = 0,\ 1 \le i \le \frac{t}{2} - 1$

$f^+(w_{2i+1}) = 0,\ 1 \le i \le \frac{t}{2} - 1$; $f^-(w_{2i+1}) = -\left[\frac{t}{2} + n + m + 1 + 2i\right],\ 1 \le i \le \frac{t}{2} - 1$

$f^+(w_t) = \frac{t}{2}$; $f^-(w_t) = -\frac{m}{2}(t+n+m)$

$f^+(u_1) = \frac{t}{2} + n + m$; $f^-(u_1) = 0$

$f^+(u_{2j}) = \frac{t}{2} - 1 + 2j,\ 1 \le j \le \frac{n-1}{2}$; $f^-(u_{2j}) = 0,\ 1 \le j \le \frac{n-1}{2}$

$f^+(u_{2j+1}) = \left[\frac{t}{2} + n + m + 1 - 2j\right],\ 1 \le j \le \frac{n-1}{2}$; $f^-(u_{2j+1}) = 0,\ 1 \le j \le \frac{n-1}{2}$

$f^+(v_{2k-1}) = \left[\frac{t}{2} + n - 2 + 2k\right],\ 1 \le k \le \frac{m}{2}$; $f^-(v_{2k-1}) = 0,\ 1 \le k \le \frac{m}{2}$

$f^+(v_{2k}) = \frac{t}{2} + n + 3 - 2k,\ 1 \le k \le \frac{m}{2}$; $f^-(v_{2k}) = 0,\ 1 \le k \le \frac{m}{2}$

Then the induced vertex labels are,

$g(u_1) = \frac{t}{2}+n+m$; $g(u_{2j}) = \frac{t-2}{2}+2j$, $1 \le j \le \frac{n-1}{2}$

$g(u_{2j+1}) = \frac{t}{2}+n+m+1-2j$, $1 \le j \le \frac{n-1}{2}$; $g(v_{2k-1}) = \frac{t}{2}+n-2+2k$, $1 \le k \le \frac{m}{2}$

$g(v_{2k}) = \frac{t}{2}+n+3-2k$, $1 \le k \le \frac{m}{2}$

Case (i): $\frac{t}{2}$ **is even**

$g(w_{2i-1}) = \frac{t}{2} + 1 - 2i$, $1 \le i \le \frac{t}{4}$; $g(w_{2i}) = \frac{t}{2} + n + m + 2i$, $1 \le i \le \frac{t}{4} - 1$

$g\left(w_{\frac{t-4}{2}+2i}\right) = 2i - 2$, $1 \le i \le \frac{t}{4} + 1$; $g\left(w_{\frac{t-2}{2}+2i}\right) = t + n + m + 1 - 2i$, $1 \le i \le \frac{t}{4}$

Case (ii): $\frac{t}{2}$ **is odd**

$g(w_{2i-1}) = \frac{t}{2}+1 - 2i$, $1 \le i \le \frac{t+2}{4}$; $g(w_{2i}) = \frac{t}{2} + n + m + 2i$, $1 \le i \le \frac{t-2}{4}$

$g\left(w_{\frac{t-2}{2}+2i}\right) = 2i - 1$, $1 \le i \le \frac{t+2}{4}$; $g\left(w_{\frac{t}{2}+2i}\right) = t + n + m - 2i$, $1 \le i \le \frac{t-2}{4}$

Clearly, $g(V) = \{0, 1, \ldots, (t + n + m - 1)\} = \{0, 1, \ldots, p - 1\}$

So, it follows that all the vertex labels are distinct and g is a bijection. Hence, $T_{t,n,m}$ is a directed edge - graceful graph if t, m are even and n is odd. The directed edge - graceful labeling of $T_{28,5,6}$ and $T_{30,5,6}$ is given in Fig. (11) and Fig. (12) respectively.

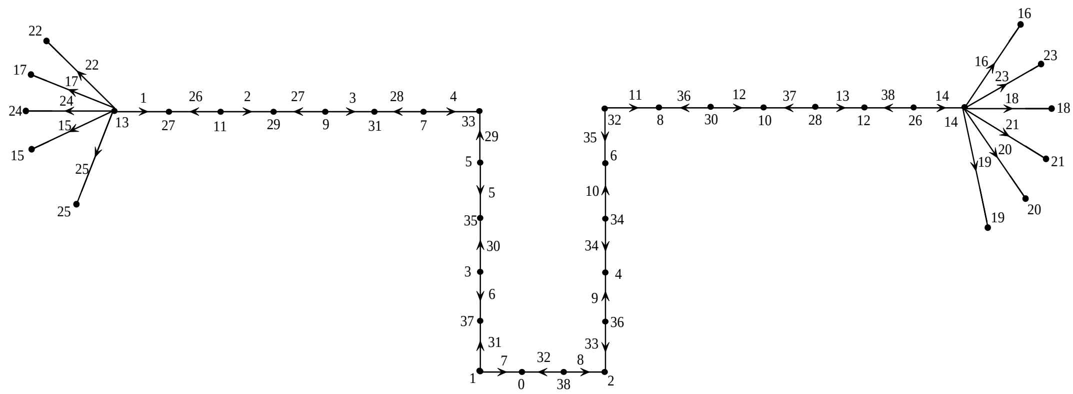

Fig (11) : $T_{28,5,6}$ with directed edge-graceful labeling

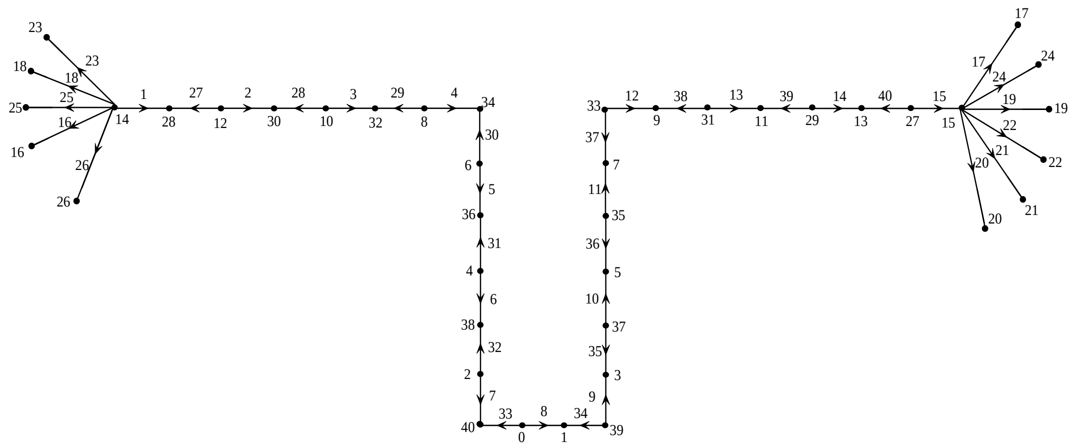

Fig (12) : $T_{30,5,6}$ with directed edge-graceful labeling

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Even Distance Closed Domination in Graph

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Abstract: *In a graph G=(V,E), a set $S \subset V(G)$ is a distance closed set of G if for each vertex $u \in S$ and for each $w \in V\text{-}S$, there exists at least one vertex $v \in S$ such that $d_{<S>}(u, v) = d_G(u, w)$. Also, a vertex subset D of V(G) is a dominating set of G if every vertex in V-D is adjacent to at least one vertex in D. Combining the above concepts, a distance closed dominating set of a graph G is defined as follows: A subset $S \subseteq V(G)$ is said to be a distance closed dominating (D.C.D) set, if <S> is distance closed and S is a dominating set. In this paper, we define a new concept of domination called even distance closed domination (E.D.C.D) and we find various bounds for these parameters and characterized the graphs, which attain these bounds.*

Keywords: *domination number, distance, eccentricity, radius, diameter, self centered graph, neighborhood, induced sub graph, distance closed dominating set, even distance closed dominating set.*

1. Introduction

Graphs discussed in this paper are connected and simple only. For a graph, let V(G) and E(G) denotes its vertex and edge set respectively. The *degree* of a vertex v in a graph G is denoted by $deg_G(v)$. The length of any shortest path between any two vertices u and v of a connected graph G is called the *distance between u and v* and it is denoted by $d_G(u, v)$. The distance between two vertices in different components of a disconnected graph is defined to be ∞. For a connected graph G, the *eccentricity* $e_G(v) = \max \{d_G(u,v): \forall u \in V(G)\}$. If there is no confusion, we simply use the notion deg (v), d(u, v) and e(v) to denote degree, distance and eccentricity respectively for the connected graph. The minimum and maximum eccentricities are the *radius* and *diameter* of G, denoted by r(G) and diam (G) respectively. If these two are equal in a graph, that graph is called *self-centered* graph with radius r and is called an *r self-centered* graph. Such graphs are 2-connected graphs. A vertex u is said to be an eccentric vertex of v in a graph G, if d (u, v)=e(v) in that graph. In general, u is called an *eccentric vertex*, if it is an eccentric vertex of some vertex. For $v \in V(G)$, the *neighborhood* $N_G(v)$ of v is the set of all vertices adjacent to v in G. The set $N_G[v]= N_G(v) \cup \{v\}$ is called the *closed neighborhood* of v. A set S of edges in a graph is said to be independent if no two of the edges in S are adjacent. An edge e=(u, v) is a *dominating edge* in a graph G if every vertex of G is adjacent to at least one of u and v. For any set S of vertices in G, the induced sub graph <S> is the maximal sub graph with vertex set S. Also, a sub graph H of G is a component

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of G if H is a maximal connected sub graph of G. The concept of distance and related properties are studied in [2], [3] and [14]. Also, the structural properties of some special class of graphs such as self centered graphs, radius critical graphs and eccentricity preserving spanning trees are studied in [4], [5], [8] and [10].

The concept of domination in graphs was introduced by Ore [13]. A set $D \subseteq V(G)$ is called dominating set of G if every vertex in V(G)-D is adjacent to some vertex in D and D is said to be a minimal dominating set if D-{v} is not a dominating set for any $v \in D$. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set. We call a set of vertices a γ-set if it is a dominating set with cardinality $\gamma(G)$.Different types of dominating sets have been studied by imposing conditions on the dominating sets. A dominating set D is called *connected* (*independent*) *dominating set* if the induced sub graph <D> is connected (independent). D is called a *total dominating set* if every vertex in V(G) is adjacent to some vertex in D. A set D is called an *efficient dominating set* of G if every vertex in V-D is adjacent to exactly one vertex in D. A set $D \subseteq V(G)$ is called a *global dominating set* if D is a dominating set of G and $\overline{G}$. A set D is called a *restrained dominating set* if every vertex in V-D is adjacent to a vertex in D and another vertex in V-D. By γ_c, γ_i, γ_t, γ_e, γ_g and γ_r, we mean the minimum cardinality of a connected dominating set, independent dominating set, total dominating set, efficient dominating set, global dominating set and restrained dominating set respectively. The list of survey of domination theory papers are in [6], [7], [12], [15], [16] and [17].

The new concepts such as distance closed sets, distance preserving sub graphs, eccentricity preserving sub graphs, super eccentric graph of a graph, pseudo geodetic graphs are introduced and structural properties of those graphs are studied in [9]. Janakiraman and Alphonse [1] introduced and studied the concept of weak convex dominating sets, which mixes the concept of dominating set and distance preserving set. Using these, structural properties of various dominating parameters are studied. Continuing the above study, the concept of distance closed dominating set was defined and the structural properties of distance closed domination in various graphs are studied in [11].

In this paper, we introduce a new dominating set called even distance closed dominating set of a graph through which we studied the properties of the graph. We find upper and lower bounds of the new domination number in terms of various already known parameters. Also, we studied several interesting properties like Nordhaus-Gaddum type results relating the graph and its complement.

2. Prior Results

The concept of distance closed set is defined and studied in the doctoral thesis of Janakiraman [9] and the concept of distance closed sets in graph theory is due to the

related concept of ideals in ring theory in algebra. The ideals in a ring are defined with respect to the multiplicative closure property with the elements of that ring. Similarly, the distance closed dominating set is defined with respect to the distance closed property and the dominating set of the graph. Thus, the distance closed dominating set of a graph G is defined as follows:

A subset $S \subseteq V(G)$ is said to be a distance closed dominating (D.C.D) set, if

(i) <S> is distance closed;

(ii) S is a dominating set.

The cardinality of a minimum D.C.D set of G is called the distance closed domination number of G and is denoted by γ_{dcl}.

Clearly from the definition, $1 \leq \gamma_{dcl} \leq p$ and graph with $\gamma_{dcl}=p$ is called a 0-distance closed dominating graph. Also, if S is a D.C.D set of G then the complement V-S need not be a D.C.D set of G. The definition and the extensive study of the above said distance closed dominating set in Graphs are studied in [11].

Following are some of the results related to the distance closed domination number of a graph presented in [11].

Theorem 2.1 [11]: If T is a tree with number of vertices $p \geq 2$, then $\gamma_{dcl}(T) = p-k+2$, where k is the number of pendant vertices in T.

Theorem 2.2 [11]: Let G be a Self centered graph of diameter 2.Then $\gamma_{dcl}(G) \leq \delta+2$.

Theorem 2.3 [11]: Let G be a graph of order p. Then

(i) $\gamma_{dcl}(G)=2$ if and only if G has at least two vertices of degree p-1.

(ii) If G has exactly a vertex of degree p-1, then $\gamma_{dcl}(G)=3$.

Theorem 2.4 [11]: Let G be a graph of order p. If G has exactly a vertex of degree p-1, then $\gamma_{dcl}(G)=3$.

Theorem 2.5 [11]: If a graph G is connected and diam(G) ≥3, then $\gamma_{dcl}(\overline{G})=4$.

Theorem 2.6 [11]: For any connected graph G such that $\overline{G}$ is also connected, $\gamma_{dcl}(G)+\gamma_{dcl}(\overline{G}) \leq p+4$, where $\gamma_{dcl}(G)$ and $\gamma_{dcl}(\overline{G})$ are the cardinality of minimal distance closed dominating set of G and $\overline{G}$ respectively.

3. Main Results:

In this paper, we define a new domination parameter namely, even distance closed domination as follows.

Even distance closed dominating sets in Graphs:

A distance closed dominating set D is said to be an even distance closed dominating set (E.D.C.D), if for any vertex $v \in V$-D, there exists $v \in D$ at even distance from u.

The cardinality of minimum even distance closed dominating set is denoted by γ_{edcl}.

Theorem 3.1: There is no graph G, with $\gamma_{edcl}(G)=2$.

Proof: Suppose that, there is a graph G having an E.D.C.D set D with $|D|=2$. Then, clearly the two vertices in D are with eccentricity 1 and they adjacent to all the vertices of V-D. Hence, every vertex in V-D is at odd distance from D and hence D cannot be an E.D.C.D set of G.

Theorem 3.2: For any tree T, $\gamma_{edcl}= \gamma_{dcl}=p-k+2$.

Proof: Proof follows from Theorem 2.1[11].

Theorem 3.3: $\gamma_{edcl}(G)=3$ if and only if G satisfies the following two properties:

(i) G has exactly one vertex with eccentricity 1;

(ii) There exists two vertices u and v such that $N(u) \cap N(v)=\{x\}$, where x is a vertex with eccentricity 1.

Proof: From the first property (i), we have $\gamma_{dcl}(G)=3$.

Let x be a vertex with eccentricity 1. If there exists two vertices u and v such that $N(u) \cap N(v)=\{x\}$ then $D=\{u,x,v\}$ forms a D.C.D set of G and every vertex in V-D is at a distance 2 from u or v or both through x. Hence D is also an E.D.C.D set of G and hence $\gamma_{edcl}(G)=3$.

Conversely, assume that $\gamma_{edcl}(G)=3$. Suppose that, if there exists $y \in N(u) \cap N(v)$, $y \neq x$ then which only means that y is adjacent to both u and v. As u and v are arbitrary, $\gamma_{edcl}(G) \geq 4$, a contradiction to $\gamma_{edcl}(G)=3$.Thus, there exists two vertices u and v such that $N(u) \cap N(v)=\{x\}$.

Proposition 3.1: For any self centered graph of diameter 2, $\gamma_{edcl} \leq \delta+2$.

Proof: Let v be a vertex of G with degree δ. Then $D= v \cup N_1(v) \cup w$, where $w \in N_2(v)$ forms a D.C.D set of G and $\gamma_{dcl} \leq \delta+2$. Also any vertex $u \in V$-D is a vertex of $N_2(v)$. This implies that $d(u,v)=2$. That is, D itself form an E.D.C.D set for G. Hence $\gamma_{edcl} \leq \delta+2$.

Remark 3.1: The above bound attained for C_5 and Petersen graph. Also, these graphs having the property that, every D.C.D set is an E.D.C.D set. Except these graphs, there are many 2-self centered graphs having this property (for example complete bipartite graph). Hence, we have the following theorem.

Theorem 3.4: If G is a 2-self centered graph and if for every $v \in V(G)$ both $<N_1(v)>$ and $<N_2(v)>$ are independent sets then, every D.C.D set of G is an E.D.C.D set of G.

Proof: Let G be a 2-self centered graph. Since, for every $v \in V(G)$ both $<N_1(v)>$ and $<N_2(v)>$ are independent, every vertex in $N_1(v)$ is adjacent to all the vertices of $N_2(v)$. Hence γ_{dcl}=4 and every D.C.D set D of G must contain at least one vertex from both $N_1(v)$ and $N_2(v)$, say $x \in N_1(v)$ and $y \in N_2(v)$. Then, every vertex in $N_2(v)$ is at a distance 2 from y and every vertex in $N_1(v)$ is at a distance 2 from x and also v is at a distance 2 from y. Hence D becomes an E.D.C.D set of G and hence every D.C.D set of G is an E.D.C.D set of G.

Theorem 3.5: If G is a 2-self centered graph, then a D.C.D set D of G is an E.D.C.D set of G, if for every vertex in $u \in V\text{-}D$, $E(u) \cap D \neq \Phi$.

Proof: Let G be a 2-self centered graph and let D be a D.C.D set of G. If D is also an E.D.C.D set of G then which only means that, every vertex in V-D is at a distance 2 from at least one vertex of D. That is, every vertex in V-D has at least one eccentric vertex in D as G is 2-self centered. Hence $E(u) \cap D \neq \Phi$.

Theorem 3.6: If G is a graph with radius 2 and diameter 3, then a D.C.D set D of G is an E.D.C.D set of G, if for every central vertex u in V-D, $E(u) \cap D \neq \Phi$.

Proof: Let G be a graph with radius 2 and diameter 3 and let D be a D.C.D set of G. If a vertex
$u \in V\text{-}D$, then we have the following cases:

Case 1: e(u)=3.

Every vertex with eccentricity 3 must be non adjacent to at least one vertex of D. For otherwise, if u is adjacent to all the vertices of D, then the eccentric node of u(say v) must be in V-D and v is non adjacent to all the vertices of D as d(u,v)=3, a contradiction to D is a D.C.D set of G. Hence u is non adjacent to at least one vertex of D and hence every vertex with eccentricity 3 must be at a even distance from at least one vertex of D.

Case 2: e(u)=2.

Suppose that, if a vertex with eccentricity 2 is adjacent to all the vertices of D then D can not be an E.D.C.D set of G. Thus, for D is also an E.D.C.D set of G if u is non adjacent to at least one vertex of D. That is, u has at least one eccentric node in D. Hence for every central vertex u in V-D, $E(u) \cap D \neq \Phi$.

Theorem 3.7: For any graph G with radius ≥2, $\gamma_{edcl}=\gamma_{dcl}$ if and only if for all minimum D.C.D set D in G, D-(D $\cap$N(u)) $\neq \Phi$, for all u$\in$V-D.

Proof: Suppose that $\gamma_{edcl}=\gamma_{dcl}$. To prove there exists a D.C.D set D in G such that for every vertex u$\in$V-D, D-(D $\cap$N(u)) $\neq \Phi$. If not, there exists no such D. Now, consider a minimal D.C.D set D of G. Then there exists a vertex u$\in$V-D such that D$\subseteq$N(u). This implies that u is adjacent to all the vertices of D. That is u is at odd distance from vertices of D. This implies that D itself can not form an E.D.C.D set for G. This is true for all minimal D.C.D set in G, which is a contradiction to $\gamma_{edcl}=\gamma_{dcl}$. Hence for every vertex u$\in$V-D, D-(D $\cap$N(u)) $\neq \Phi$.

Conversely, if there exists a minimal D.C.D set D such that D-(D $\cap$N(u)) $\neq \Phi$, for every vertex u$\in$V-D. Since D is a dominating set, every vertex u$\in$V-D is dominated by some vertex v and also $\gamma_{dcl}\geq 4$, that is |D| ≥4 and u is not adjacent to all the vertices of D. Therefore, D itself forms an E.D.C.D set for G. Hence $\gamma_{edcl}=\gamma_{dcl}$.

Theorem 3.8: For any graph G, if $\gamma_{edcl}\neq\gamma_{dcl}$, then diameter of G is less than or equal to 4.

Proof: Let D be a minimal D.C.D set of G. If $\gamma_{edcl}\neq\gamma_{dcl}$ then by previous proposition, there exists a vertex u$\in$V-D such that all the vertices of D are adjacent to u. ---------------- (1)

Let x and y be any two non adjacent vertices of G

Case 1: If x and y are in D.

Then from (1), x and y are adjacent to u. This implies that d(x,y)=2.

Case 2: If x$\in$D and y$\in$V-D.

Since D is a dominating set, there exists a vertex v$\in$D such that y is adjacent to v and also from previous case d(x,v)≤ 2. This implies that d(x,y)≤ 3.

Case 3: If both x,y are in V-D.

Since D is a dominating set, x and y are dominated by some vertices x^1, y^1 in D respectively. From case 1, d(x^1,y^1)≤ 2. This implies that d(x,y)≤ 4. Hence the proof.

Theorem 3.9: If G is a graph with radius≥3, then every D.C.D set of G is an E.D.C.D set of G.

Proof: Let G be a graph with radius≥3 and let D be a D.C.D set of G.

Claim: D is also an E.D.C.D set of G.

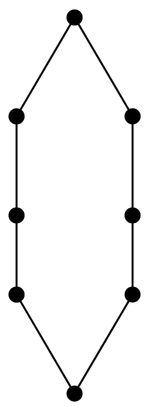
Suppose that, if a vertex u$\in$V-D is adjacent to all the vertices of D then e(u)=2(as every vertex of V-D is adjacent to at least one vertex of D), a contradiction to radius of G ≥3. Hence u is not adjacent to at least one vertex of D and hence D is also an E.D.C.D set of G.

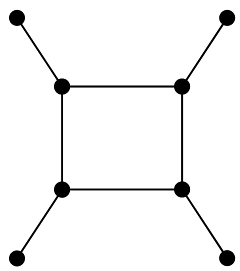
Theorem 3.10: For any graph G with diameter greater than or equal to 3, $\gamma_{edcl}(G)+\gamma_{edcl}(\overline{G})\leq p+4$.

Proof: Let u and v be two vertices of G with distance greater than or equal to 3. Then clearly the edge uv in $\overline{G}$ will form a dominating set. Also the set {x,u,v,y} forms a D.C.D set of $\overline{G}$ where $x\in N_1(v)$ and $y\in N_1(u)$. In $\overline{G}$, every vertex of $N_1(u)$(in G) is at a distance 2 from u and every vertex of $N_1(v)$(in G) is at a distance 2 from v. Also, all the vertices which are not in $N_1[u]\cup N_1[v]$(in G) must be adjacent to both u and v in $\overline{G}$. Therefore, those vertices are at a distance 2 from both x and y in $\overline{G}$. Hence {x,u,v,y} itself forms an E.D.C.D set of $\overline{G}$ and hence $\gamma_{edcl}(\overline{G})= 4$.

Therefore, $\gamma_{edcl}(G)+\gamma_{edcl}(\overline{G})\leq p+4$.

Remark 3.2: The above bound is sharp and attainable. For example





For the above graphs, $\gamma_{edcl}(G)=8$ and $\gamma_{edcl}(\overline{G})=4$ and hence $\gamma_{edcl}(G)+\gamma_{edcl}(\overline{G})=12=p+4$.

That is, for any even cycles, paths and ciliates, the bound is sharp.

Theorem 3.11: Let G and $\overline{G}$ be self centered graphs of diameter 2 then, $\gamma_{edcl}(G)+\gamma_{edcl}(\overline{G})\leq p+3$.

Proof: We have $\gamma_{edcl}(G)\leq \delta+2$.

Thus, $\gamma_{edcl}(G)+\gamma_{edcl}(\overline{G}) \leq\delta(G)+2+\delta(\overline{G})+2$

$= \delta(G)+2+\Delta(G)+2$

$= \delta(G)+\Delta(G)+4$

$=p-1+4$

$=p+3$

Hence, $\gamma_{edcl}(G)+\gamma_{edcl}(\overline{G})\leq p+3$.

Remark 3.3: This bound is sharp and attainable for the graph C_5.

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K- Equitable Labeling of Graphs

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Abstract: *Cahit introduced the k- equitable labeling as a generalization of graceful labeling. In this paper, we study on k- equitable labeling and we prove that the graph* P_n^+ *is k- equitable for all k, n and the graph* C_n^+ *is k-equitable if* $n \neq \frac{k}{2}(\text{mod } k)$ *or* $k \neq 2n+1$.

1. Introduction

Cahit has introduced a variation of both graceful and harmonious labelings[1 – 2]. Let f be a function from the vertices of G to {0,1} and for each edge xy assign the label $|f(x) - f(y)|$ and call f a cordial labeling of G if the number of vertices labeled 0 and the number of vertices labeled 1 differ at most by 1 and the number of edges labeled 0 and the number of edges labeled 1 differ at most by 1.

In 1990, Cahit proposed the idea of distributing the vertex and edge labels among {0,1,……..k-1}as evenly as possible to obtain a generalization of graceful labeling as follows: For a graph G (V,E) and a positive integer k, assign vertex labels from {0,1, ……,k-1} so that when the edge labels introduced by the absolute value of the difference of the vertex labels, the number of vertices labeled with i and the number of vertices labeled with j differ by at most one and the number of edges labeled with i and the number of edges labeled with j differ by atmost one. Cahit has called a graph with such an assignment of labels k-equitable.

A graph G (V, E) is graceful if and only if it is $|E(G)| + 1$ – equitable. G is cordial if and only if it is 2-equitable. Szaniszlo has proved that P_n is k-equitable for all k[4]. In this paper, we study on k- equitable labeling and we prove that the graph P_n^+ is k- equitable for all k,n. For an extensive survey on graph labeling we refer to Gallian[3].

2. Main Result

Theorem 2.1 : If P_n is the path on *n* vertices, the graph P_n^+ is k-equitable for any k, $n \in N$.

Proof : Let Pn be the path $v_1v_2….v_n$ and let $v'_1,v'_2….v'_n$ be the pendant vertices adjacent to $v_1,v_2,…,v_n$ respectively in P_n^+. Let k be a given positive integer.

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Let n = mk + t where $1 \le t \le k$.

Define a map f: $V(P_n^+) \longrightarrow \{0,1,2,\ldots.k-1\}$ as

a) If k is even,

$$\left.\begin{array}{l} f(v'_{2i+1}) = 2i \\ f(v_{2i+1}) = k-2i-1 \end{array}\right\} \text{for } i = 0,1,2,\ldots, \frac{k}{2}-1$$

$$\left.\begin{array}{l} f(v'_{2i}) = k-2i \\ f(v_{2i}) = 2i-1 \end{array}\right\} \text{for } i = 0,1,2,\ldots, \frac{k}{2}$$

$f(v_{kl+i}) = f(v_i)$; $f(v'_{kl+i}) = f(v'_i)$ for all l= 1,2,…..,m, i = 1,2,….k if $l \neq$ m

i = 1, 2….,t if l =m

b) If k is odd,

$$\left.\begin{array}{l} f(v'_{2i+1}) = 2i \\ f(v_{2i+1}) = k-2i-1 \end{array}\right\} \text{for } i = 0,1,2,\ldots, \frac{k-1}{2}$$

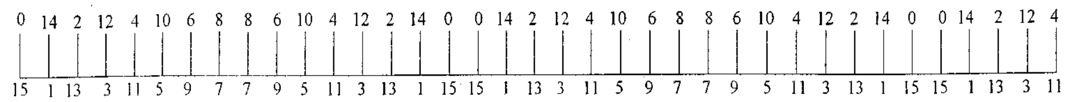
$$\left.\begin{array}{l} f(v'_{2i}) = k-2i \\ f(v_{2i}) = 2i-1 \end{array}\right\} \text{for } i = ,1,2,\ldots, \frac{k-1}{2}$$

$$\left.\begin{array}{l} f(v'_{k+j}) = f(v_{k-j+1}) \\ f(v_{k+j}) = f(v_{k-j+1}) \end{array}\right\} \text{for } j = 1,2,\ldots,k$$

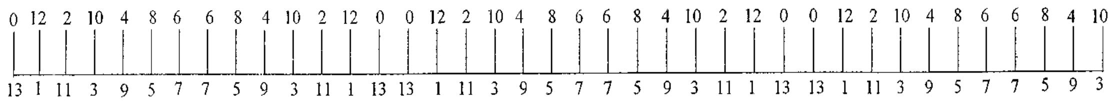
$$\left.\begin{array}{l} f(v'_{2kl+i}) = f(v_i) \\ f(v_{2kl+i}) = f(v_i) \end{array}\right\} \text{for } l = 1,2\ldots.., \left[\frac{n}{2k}\right] \text{; and}$$

for i = 1,2,….2k. if $l \neq \left[\frac{n}{2k}\right]$ and i = 1,2,…. (n-$\left[\frac{n}{2k}\right]$ 2k) if $l = \left[\frac{n}{2k}\right]$

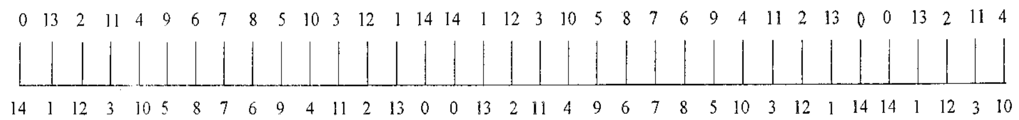
One can verify that the number of vertices labelled with i $V_f(i)$ and the number of vertices labelled with j $V_f(j)$ and the number of edges labelled with i $e_f(i)$ and the number of edges labelled with j differs by at most one for all I and j. Thus the graph P_n^+ is K-equitable for all n and k. K- equitable labeling for P_{37}^+, P_{39}^+ and P_{35}^+ are shown in Figure2.1.



An 16-Equitable labeling for P_{37}^+



An 14-Equitable labeling for P_{39}^+



An 15-Equitable labeling for P_{35}^+

Fig 2.1

Theorem 2.2 : Let n and k be positive integers such that $n \neq \frac{k}{2}$ (mod k) or $k \neq 2n+1$, then C_n^+ is k- equitable.

Proof: Let G = (V,E) be the graph C_n^+. Let $v_1, v_2, \ldots\ldots v_n$ be the cycle C_n in G and let for each $i (1 \leq i \leq n)$, v_i'be the vertex of degree one and adjacent to the vertex v_i.

First we assume that $k < n$.

Let $n = mk + r$, where $0 \leq r < k$ Define $f : V(G) \longrightarrow \{0,1,2,\ldots,k-1\}$ as follows:

$$f(v_{2kl+j}) = \begin{cases} i-1 \text{ if } i \text{ is odd and } 1 \leq i \leq k; l = 0,1,\ldots,\left[\frac{m}{2}\right] \\ k-1 \text{ if } i \text{ is even and } 1 \leq i \leq k; l = 0,1,\ldots,\left[\frac{m}{2}\right] \end{cases}$$

$$f\left(v_{(2l+1)k+i}\right) = f(v_{k-i+1}) \text{ for } l = 0,1,\ldots, \left[\frac{m}{2}\right] \text{ and } i = 1,\ldots,k.$$

$f(v'_i) = (k-1)-f(v_i)$ for all $1 \leq i \leq n$.

We consider various cases and in each case we define a k- equitable labeling $g: V(G) \longrightarrow \{0,1,\ldots,k-1\}$ by modifying the function f.

Case (i) Let k be even.

Subcase (i) Let r =1

Take $g\left(v_{mk+1}\right) = k-1$; $g(v'_1) = 0$; $g\left(v'_{mk+1}\right) = 1$ and

$g(v_i) = f(v_i)$ for all $i \le mk$. $g(v'_i) = f(v'_i)$ for all $1 < i \le mk$

Thus g is a k-equitable labeling for C^{+}_{mk+1} (k is even).

(An **14 - equitable labeling for** C^{+}_{29} **is** is illustrated in the figure 2.2).

Subcase (ii) r is odd $1 < r \le \dfrac{k+1}{3}$:

If $r \le \dfrac{k+1}{3}$ we have $r-1 \le k-2r$ In this case we take g = f, without any modification.

(ie, We define g as g(u) = f(u) for all $u \in V(G)$)

(An **16 - equitable labeling for** C^{+}_{37} is shown in figure 2.3)

Subcase(iii) r is odd, $\dfrac{k-1}{3} < r < \dfrac{k}{2}$ and $\dfrac{k-r+1}{2}$ is odd

$$g(v_{mk+j}) = \begin{cases} j & \text{if } j = \dfrac{k-r+1}{2}, \dfrac{k-r+1}{2}+2, \dots, r-2 \\ r+1 & \text{if } j = r \end{cases}$$

$$g(v'_{mk+j}) = \left\{ j \quad \text{if } j = \dfrac{k-r+1}{2}+1, \dfrac{k-r+1}{2}+3, \dots, r-1 \right.$$

and $g(v_i) = f(v_i)$ if $i \ne mk+j$ where $j \in \left\{\dfrac{k-r+1}{2}, \dfrac{k-r+1}{2}+2, \dots, r-2, r\right\}$

$g(v'_i) = f(v'_i)$ if $i \ne mk+j$ where $j \in \left\{\dfrac{k-r+1}{2}+1, \dfrac{k-r+1}{2}+3, \dots, r-1\right\}$

(A **24 – equitable labeling for** C^{+}_{35} is shown in the Figure 2.4)

Subcase(iv) r is odd $\dfrac{k-1}{3} < r < \dfrac{k}{2}$ and $\dfrac{k-r+1}{2}$ is even i.e., (r-1 = k mod 4)

Let $g(v'_{mk+j}) = j$ if j is even and $j \ge \dfrac{k-r+1}{2}$

$g(v_{mk+j}) = j$ if j is odd and $j \ge \dfrac{k-r+1}{2}$

and $g(v_i) = f(v_i)$ for all other v_i and $v'i$ $g(v'_i) = f(v'_i)$ for all other vertices

(A **26 – equitable labeling for** C^{+}_{37} is shown in the Figure 2.5).

Subcase (v) $\dfrac{k}{2} < r < k$ and r is odd

In this case we take g = f .

(In the figure 2.6 , a **26 equitable labeling for** C^{+}_{45} is shown)

Now we consider the cases when r is even.

Subcase (vi) Let r = 2

(As $r \neq \frac{k}{2}$ and k is even we have k =2 or $k \geq 6$ but $k \neq 2$, as $r < k$).

Assume that $k \geq 6$

We define g as

$$g(v_{mk+1}) = 0; \qquad g(v_{mk+2}) = k-1$$

$$g(v'_{mk+1}) = 2 ; \qquad g(v'_{mk+2}) = 1 \text{ and } \quad g(v'_1) = 3$$

and g(u) = f(u) for all other vertices u.

(A **18 - equitable labeling for** C_{38}^{+} is shown in the figure 2.7)

Subcase (vii) r is even and $2 < r \leq \frac{k+2}{3}$.

Let $g(v_{mk+r}) = r$ and g(u) = f(u) for all $u \in V(G) - \{v_n\}$

(See the figure 2.8 ,for a **18- equitable labeling for** C_{40}^{+} .)

Subcase (viii) r is even and $\frac{k+2}{3} < r < \frac{k}{2}$

If r = 0 (mod 4) we define g as follows:

$g(v_{mk+j}) = j$ for all $j = \frac{r}{2}+1 , \frac{r}{2}+3 ,\ldots,r$

$g(v'_{mk+j}) = j$ for all $j = \frac{r}{2}+2 , \frac{r}{2}+4 ,\ldots,r-1$

g(u) = f(u) for all other $u \in V(G)$

if r =2 (mod 4) we define g as

$g(v_{mk+j}) = j$ for all $j = \frac{r}{2}+1 , \frac{r}{2}+3 ,\ldots,r$

$g(v'_{mk+j}) = j$ for all $j = \frac{r}{2}+2 , \frac{r}{2}+4 ,\ldots,r-1$

$g(v_n) = k-r-1$ and g(u) = f(u) for all other vertices $u \in V(G)$

(In the figure 2.9, a **28 - equitable labeling for** C_{40}^{+} is shown.)

Subcase(ix) r is even and $\frac{k}{2} < r < \frac{2k+1}{3}$

In this case we make no changes in f and we take g = f

(In the figure 2.10 a **28 - equitable labeling for** C_{44}^{+}))

Subcase(x) r is even and $\frac{2k+1}{3} < r < k$

If r = 0 (mod 4) define

$g(v_{mk+j}) = j$ for all $j = \frac{r}{2}+1 , \frac{r}{2}+3 ,\ldots,r-1$

$g\left(v'_{mk+j}\right) = j$ for all $j = \frac{r}{2}+2, \frac{r}{2}+4, \ldots, r$. $g(u) = f(u)$ for all other vertices u

if r =2 (mod 4) define

$g\left(v_{mk+j}\right) = j$ for all $j = \frac{r}{2}+2, \frac{r}{2}+4, \ldots, r-1$

$g\left(v'_{mk+j}\right) = j$ for all $j = \frac{r}{2}+1, \frac{r}{2}+3, \ldots, r$; $g\left(v_n\right) = k-r-1$

and $g(u) = f(u)$ for all other vertices $u \in V(G)$.

Case (ii) Let k be odd.

Subcase (i) Let r = 1 i.e., n = mk+1

If m is odd define g as follows: $g\left(v'_n\right) = 1$, and $g(u) = f(u)$ for all $u \neq v'n \in V\left(c_n^+\right)$.

if m is even, then define the map g as follows: $g\left(v_n\right) = 0$, $g\left(v'_n\right) = k-1$; $g(u) = f(u)$ for all $u \neq$, v_n, v'_n

Subcase (ii) r is odd and $1 < r < \frac{k+2}{3}$.

If m is odd, then define $g\left(v_n\right) = r-1$; $g\left(v'_n\right) = k-r$ and $g(u) = f(u)$ for all other u .

If m is even we define $g(u) = f(u)$ for all $u \in V\left(C_n^+\right)$

Subcase (iii) Let r = 2

If m is odd, we define g as follows: $g\left(v_{n-1}\right) = k-1$; $g\left(v_n\right) = 1$

$g\left(v'_{n-1}\right) = 0$; $g(P_n^+) = k-2$ and $g(u) = f(u)$ for all $u \in V(G)$.

If m is even, we define g as follows:

$g\left(v_n\right) = (k-2)$; $g\left(v_{n-1}\right) = k-1$, $g\left(v'_n\right) = 1$; $g\left(v'_{n-1}\right) = k-1$; $g\left(v'_{n-2}\right) = 0$

and $g(u) = f(u)$ for all $u \in V(G)$.

Subcase (iv) Let r be even and $2 < r < \frac{k+2}{3}$.

If m is odd, take g = f , i.e., $g(u) = f(u)$ for all $u \in V(G)$.

If m is even, we define g as follows: $g\left(v_n\right) = r-1$; $g\left(v'_n\right) = k-r$ and

$g(u) = f(u)$ for all other $u \in V\left(C_n^+\right)$

Subcase (v) r is odd, $\frac{k+2}{3} \leq r < \frac{k}{2}$.

Let m be odd. If r-1 = 0 (mod 4) We define g as follows:

$g\left(v'_{mk+j}\right) = j$ for $j = \frac{r+1}{2}, \frac{r+1}{2}+2, \ldots, r$ and $g\left(v_{mk+j}\right) = j$ for $j = \frac{r+1}{2}+1, \frac{r+1}{2}+3, \ldots, r-1$. and $g(u) = f(u)$ for all other $u \in V(G)$.

If r-1 = 2 (mod 4) we define g as follows:

$g\left(v_{mk+j}\right)$ = k-j-1 for j = $\frac{r+3}{2}$, $\frac{r+3}{2}$+2 ,…..,r.

$g\left(v'_{mk+j}\right)$ = k-j-1 for j = $\frac{r+3}{2}$+1 , $\frac{r+3}{2}$+3,…..,r-1 and g(u) = f(u) for all other u.

Let m be even, r odd and $\frac{k-r+2}{2}$ be odd.

Define g as follows:

$g\left(v'_{mk+j}\right)$ = k-j-1 , for all odd j ≥ $\frac{k-r+2}{2}$.

$g\left(v_{mk+j}\right)$ = k-j-1, for all even j ≥ $\frac{k-r+2}{2}$ and g(u)=f(u) for all other u.

If m is even ,r odd and $\frac{k-r+2}{2}$ is even ,define as follows:

$g\left(v'_{mk+j}\right)$ = k-j-1 for j = $\frac{k-r}{2}$, $\frac{k-r}{2}$+2 ,…..,r.

$g\left(v_{mk+j}\right)$ = k-j-1 for j = $\frac{k-r}{2}$+1 , $\frac{k-r}{2}$+3 ,…..,r-1.

$g\left(v_{mk+j}\right)$ = j-2 for j = $\frac{k-r}{2}$.

$g\left(v'_{mk+j}\right)$ = j for j = $\frac{k-r}{2}$ -1.and g(u) = f(u) for all other u $\in v\left(C_n^+\right)$.

Subcase (vi): Let $\frac{k}{2} < r \le$ k-1.

We define the map g as follows: If both m and r are odd, let

$$g(v_{mk+j}) = \begin{cases} j-1 \text{ for all odd } j \ge \frac{k+1}{2}+1 \\ k-j \text{ for all even } j \ge \frac{k+1}{2}+1 \end{cases}$$

$$g(v'_{mk+j}) = \begin{cases} k-j \text{ for all odd } j \ge \frac{k+1}{2}+1 \\ j-1 \text{ for all even } j \ge \frac{k+1}{2}+1 \end{cases}$$

and g(u) = f(u) for all other vertices u .

(a) If both m and r are even, let

$$g(v_{mk+j}) = \begin{cases} j-1 \text{ for all even } j \geq \dfrac{k+1}{2}+1 \\ k-j \text{ for all odd } j \geq \dfrac{k+1}{2}+1 \end{cases}$$

$$g(v'_{mk+j}) = \begin{cases} k-j \text{ for all even } j \geq \dfrac{k+1}{2}+1 \\ j-1 \text{ for all odd } j \geq \dfrac{k+1}{2}+1 \end{cases}$$

and g(u) = f(u) for all other vertices u.

(b) If one of m and r is even and the other is odd, let

$g(u) = f(u)$ for all $u \in V(G)$.

In all the above cases one can verify that $V_f(i)$ and $e_f(i)$ differs by at most one for all i. Thus the graph C_n^+ is K-equitable for all $n \neq \dfrac{k}{2}$ (mod k) or $k \neq 2n+1$, .

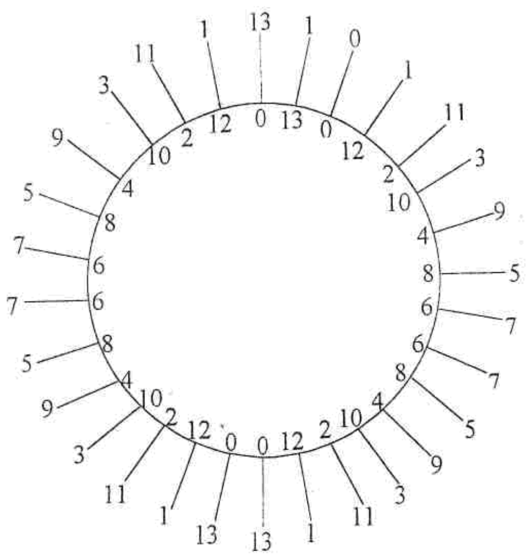


Fig 2.2 : A 14-equitable labelling for C_{29}^+

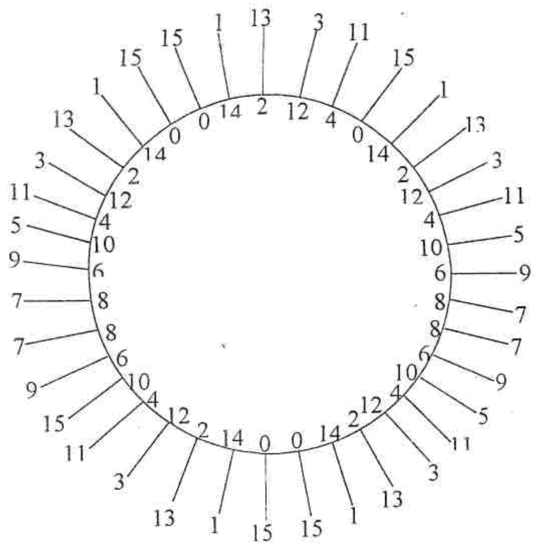


Fig 2.3 : A 16-equitable labelling for C_{37}^+

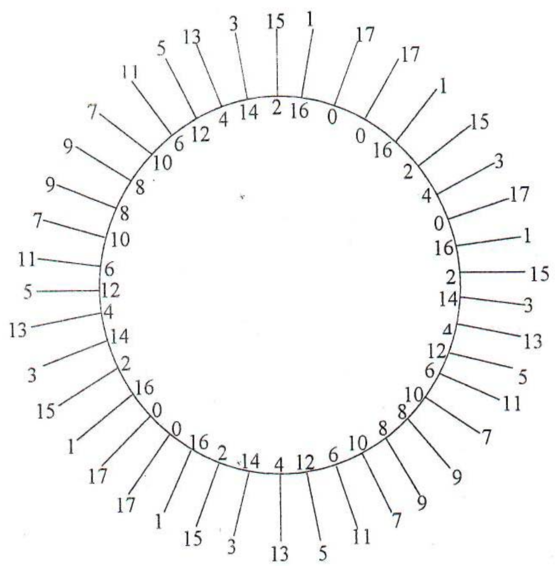


Fig 2.4 : A 24-equitable labelling for C_{35}^{+}

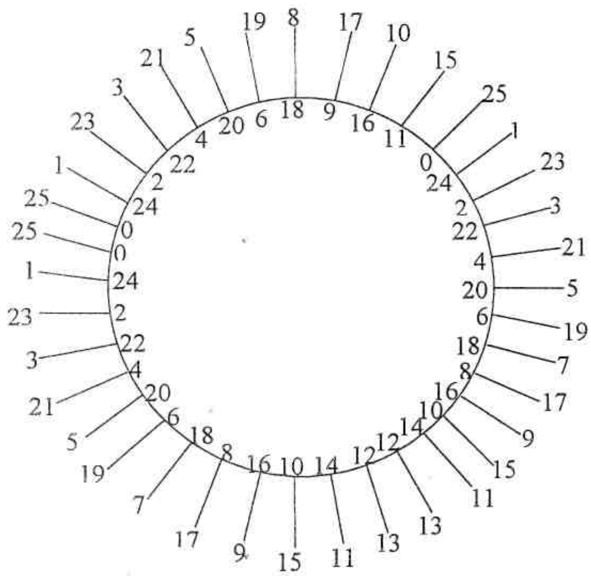


Fig 2.5 : A 26-equitable labelling for C_{37}^{+}

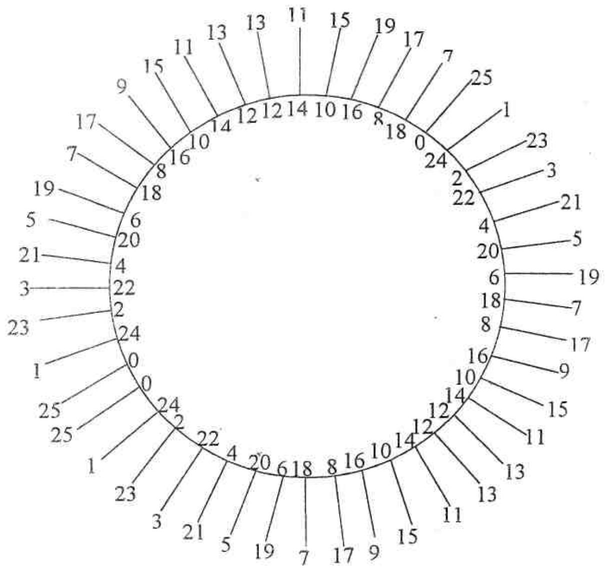


Fig 2.6 : A 26-equitable labelling for C_{45}^{+}

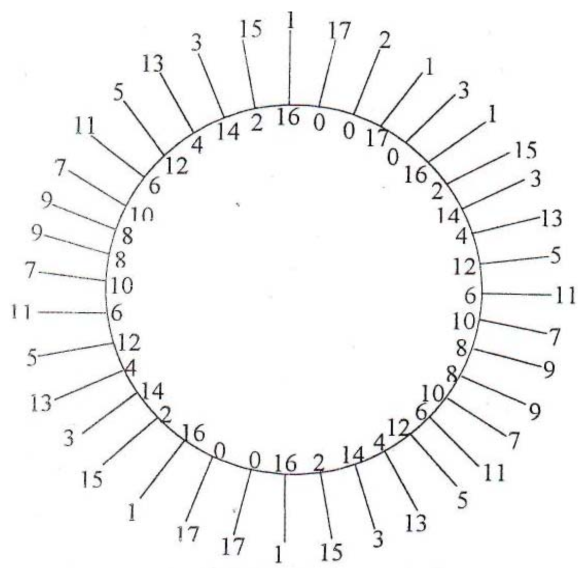


Fig 2.7 : An 18-equitable labelling for C_{38}^{+}

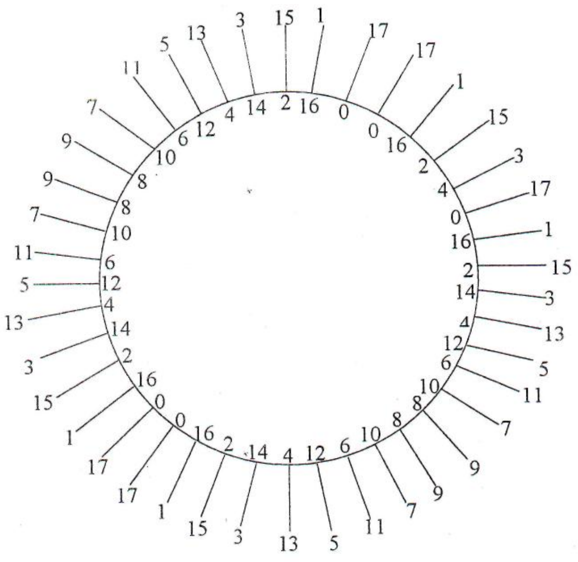


Fig 2.8 : A 18-equitable labelling for C_{40}^{+}

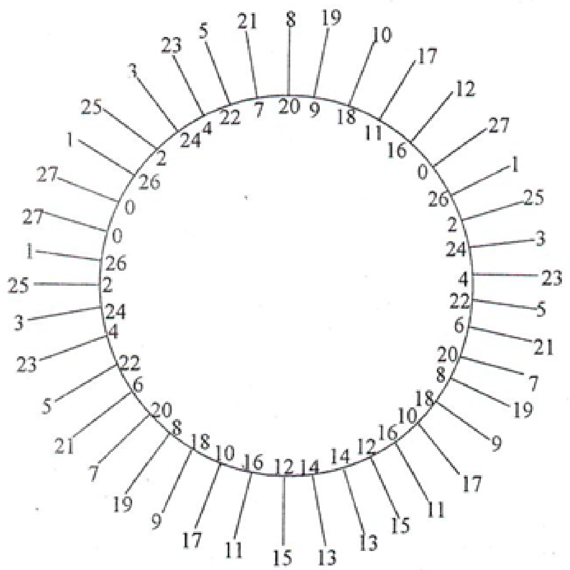


Fig 2.9 : A 28-equitable labelling for C_{40}^{+}

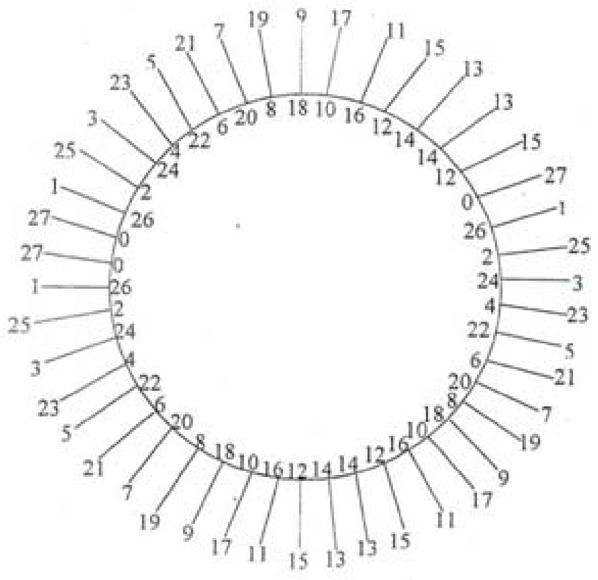


Fig 2.10 : A 28-equitable labelling for C_{44}^{+}

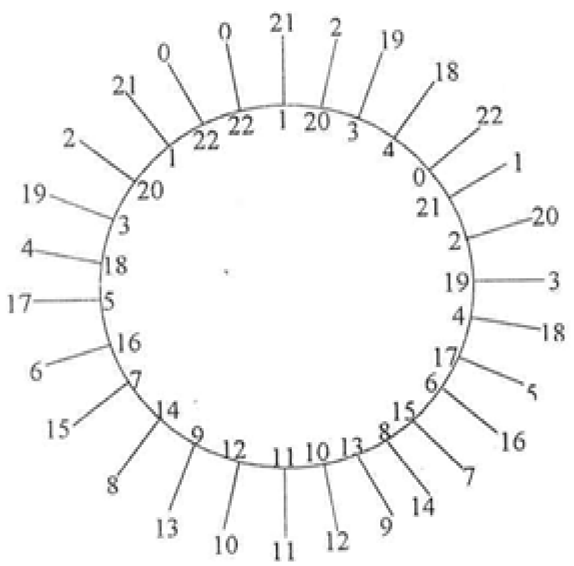


Fig 2.11 : A 23-equitable labelling for C_{28}^{+}

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